Exit Equilibrium: Towards Understanding Voluntary Participation in Security Games

Parinaz Naghizadeh and Mingyan Liu
Department of EECS, University of Michigan, Ann Arbor
Email: {naghizad, mingyan}@umich.edu

***Abstract*—In a system of interdependent users, the security of an entity is affected not only by that user's effort towards securing her system, but also by the security decisions of other users. The provision of security in such environment is modeled as a public good provision problem, and is referred to as a security game. In this paper, we propose the notion of exit equilibrium to study users' voluntary participation in mechanisms for provision of non-excludable public goods. We show the fundamental result that, due to the non-excludable nature of security, there exists no reliable mechanism which can incentivize the socially optimal investment profile, while ensuring voluntary participation and maintaining a weakly balanced budget, for all instances of security games. To better understand the features of the games that lead to this result, we consider the class of weighted effort games, and apply the two well-known Pivotal (VCG) and Externality mechanisms. Through analysis and simulation, we identify the effects of several features of the problem environment, including diversity in user types, multiplicity of exit equilibria, and users' self-dependence levels, on the performance of these mechanisms.**

I. Introduction

Attempts to improve the state of cyber security have been on the rise over the past years. The need for these initiatives is evident from studies indicating a 95% increase in the average cost of cyber crime per company from 2010 to 2014 [1], as well as the rising number of high profile attacks, including those on Target, JP Morgan, and Sony Pictures, in 2014. Adoption of better security practices by individual entities in this landscape will not only provide direct protection against attacks, but can further provide *positive externalities* to other interacting users, by reducing the probability of attacks initiating or propagating from protected entities. Consequently, the provision of security in such interconnected systems is viewed as a *public good* provision problem. In particular, *security games* have been proposed to study the level of effort exerted by strategic users in such environments, as well as to design mechanisms for incentivizing the adoption of better security practices, see [2]–[7]. Our focus in the current paper is on the use of monetary payments/rewards to incentivize *socially optimal* (SO) security behavior, i.e.; those minimizing the collective cost of security.

Aside from inducing optimal behavior, incentive mechanisms are often designed so as to maintain a *weakly balanced budget (WBB)* and ensure *voluntary participation (VP)* by all users. The weak budget balance requirement states that the designer of the mechanism prefers to redistribute users' payments as rewards, and ideally to either retain a surplus as profit or at least to not sustain losses. Otherwise, the designer would need to spend external resources to achieve social optimality. The voluntary participation constraint on the other hand ensures that all users voluntarily take part in the proposed mechanism, preferring its outcome to that attained had they unilaterally opted out. A user's decision when contemplating participation in an incentive mechanism is dependent not only on the structure of the induced game, but also on the options available when staying out. The latter is what sets the study of incentive mechanisms for security games (as well as other *non-excludable* goods) apart from other (excludable) public good problems.

To elaborate on this underlying difference, note that due to the non-excludable nature of security, although the mechanism optimizes the investments in a way that participating users are exposed to lower risks, those who stay out of the mechanism can benefit from the externalities of such improved state of security as well, while choosing their own investment levels independently. We therefore introduce the notion of *exit equilibrium*, to describe users' outside options from mechanisms for the provision of such non-excludable goods; at this equilibrium, a user unilaterally opts out of the proposed mechanism, and best-responds to the remaining users who continue participating.

In contrast to security, with excludable public goods, users' willingness to participate is determined by the change in their utilities when contributing and receiving the good, as compared to receiving *no allocation at all*. This means that the designer has the ability to collect more taxes and require a higher level of contribution when providing an excludable good. As a result, tax-based mechanisms, such as the Externality mechanism (e.g. [8]) and the Pivotal mechanism (e.g. [9]), can be designed so as to incentivize the socially optimal provision of an excludable good, guarantee voluntary participation, and maintain weak budget balance.

However, in this paper we first show the fundamental result that, with non-excludable goods, there is no tax-based mechanism that can achieve social optimality, voluntary participation, and weak budget balance simultaneously in all instances of the game; i.e., without further information about the network structure and users' utility functions. We show how this conflict can emerge in many classes of security games through two general families of counter-examples: first in network structures with a star topology, and next by considering the commonly studied weakest link model for users' risk functions. We identify two classes of users: *main investors*, who receive a reward in return for improving their investment levels (from which themselves and other users benefit), and *free-riders*, who pay a tax to benefit from a more secure environment. Our counter-examples highlight how users from either class may decide to opt out of the mechanism.

We then further elaborate on this result by examining the specific class of *weighted effort* games. This interdependence model is of particular interest as it can capture varying degrees and possible asymmetries in the influence of users' security decisions on one another. We evaluate the effects of: (i) increasing users' self-dependence (equivalently, decreasing their interdependence), (ii) having two diverse communities of self-dependent and reliant users, and (iii) the presence of a single dominant user, on the performance of two well-known tax-based incentive mechanisms, namely the Pivotal and Externality mechanisms.

Our analysis and simulations[1] lead to the following insights on the features of the problem environment affecting the mechanisms' performance. We show that participation incentives may be satisfied when exit equilibria are less beneficial to the outliers (e.g., require a free-rider to become a main investor). A similar result holds when multiple exit equilibria are possible, and participating users can coordinate on the least beneficial equilibrium for the outlier. We further highlight how increased self-dependence can aid the performance of both incentive mechanisms, as users themselves find it more beneficial to improve their investments. In addition, we identify restricted families of positive instances, in which either of our mechanisms can simultaneously guarantee SO, WBB, and VP. One such instance emerges when the game consists of two communities of self-dependent and reliant users. In such scenario, an appropriate design of taxes can lead to a *transfer of funds* from reliant users to the self-dependent community in return for a more secure environment. Another instance is one where users coordinate to *exchange favors*, increasing their investments in return for increased effort by others.

Research contributions: Our main contributions in this paper can be summarized as follows:

• We introduce the notion of exit equilibrium to describe strategic users' outside options in mechanisms for incentivizing the adoption of optimal security practices. Our work is hence the first in the literature to formalize the study of voluntary participation in security games.

• We show the fundamental impossibility of simultaneously guaranteeing social optimality, voluntary participation, and weak budget balance in all instances of security games. By comparing this finding to existing possibility results (see Section V), our work highlights the crucial effect of users' outside options on the design of any mechanism for improving users' security behavior. Our insights are also applicable to other problems concerning the provision of non-excludable public goods over social and economic networks (Section V).

• We identify several features of an environment that can affect the performance of incentive mechanisms for security games, as well as restricted families of positive instances. These findings can guide the selection of a mechanism given additional information about the problem environment.

[1] Due to space considerations, we present the majority of our formal analysis in the online appendix [10]. Our analysis reflects that, even within these restricted subclasses, a full characterization of exit equilibria, and consequently the study of incentive mechanisms, can become increasingly complex, making numerical simulations an attractive alternative.

II. Security games: Model and Preliminaries

A. Model

Consider a network of N interdependent users; these can be computers on a network, different divisions within a large company, or different sectors of a country's economy. In all these scenarios, each user i independently chooses to exert effort towards securing her system, consequently achieving the *level of security, investment* or *effort* $x_i \in \mathbb{R}_{\geq 0}$. This decision not only affects the security of user i's system, but has an effect on some or all other users $j \neq i$'s security. Let $\mathbf{x} := \{x_1, x_2, \ldots, x_N\}$ denote the *state of security* of the system; i.e., the profile of security levels of all N users.

A user i's *(security) cost function* at a state of security $\mathbf{x}$ is given by:

$$g_i(\mathbf{x}) := f_i(\mathbf{x}) + h_i(x_i) \ . \tag{1}$$

Here, $f_i(\cdot) : \mathbb{R}^N_{\geq 0} \to \mathbb{R}_{\geq 0}$ denotes user i's *risk function*, determining the expected amount of assets user i has subject to loss at a state of security $\mathbf{x}$. The second term, $h_i(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, denotes the *investment cost function* of user i; it determines the expenditure by user i required to implement a level of security x_i. We make the following assumptions on the risk and cost functions.

Assumption 1: For all users i, $f_i(\cdot)$ is continuous, differentiable, decreasing, and strictly convex, in all arguments x_j.

Intuitively, the decreasing nature of this function in arguments $x_j, j \neq i$, models the positive externality of users' security decisions on one another. The convexity on the other hand implies that the effectiveness of security measures in preventing attacks (or the marginal benefit) is overall decreasing, as none of the available security measures can guarantee the prevention of all possible attacks.

Assumption 2: For all users i, $h_i(\cdot)$ is continuous, differentiable, strictly increasing, and strictly convex.

Intuitively, the assumption of convexity entails that while implementing basic security measures is relatively cheap (e.g. limiting administrative privileges), protection becomes increasingly costly as its effectiveness increases (e.g. implementing intrusion detection systems).

Formally, a *security game* refers to the simultaneous move full information game among the N utility maximizing users with utility functions $u_i(\mathbf{x}) = -g_i(\mathbf{x})$, where each user's action is the choice of a level of security $x_i \geq 0$.

B. Nash equilibrium, social optimality, and exit equilibrium

The levels of security implemented by users at different equilibria of security games have been extensively studied in the literature; we refer the interested reader to [2] for a survey. Throughout this literature, the most commonly studied equilibrium concept is that of *Nash equilibrium (NE)*; i.e., the state of security at which each user independently chooses a security level that minimizes her costs, given the security levels selected by other users.

It is also common in the literature to measure the (sub-)optimality of these Nash equilibria by comparing them to the *socially optimal (SO)* investments. Formally, the socially optimal investment levels $\mathbf{x}^*$ are those maximizing the total welfare, or equivalently, minimizing the sum of all users' costs; i.e.,

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\succeq 0} \sum_{i=1}^{N} g_i(\mathbf{x}). \tag{2}$$

A comparison of the Nash equilibria and the socially optimal solution often reveals an under-investment is security by users. The existing literature has proposed mechanisms for decreasing this inefficiency gap, by either incentivizing or dictating improved security investments [2].

Our focus in the present paper is on regulating mechanisms that use monetary taxation to incentivize socially optimal security behavior in security games. Such mechanisms incentivize optimal behavior by assessing a tax t_i to each participating user i; this tax may be positive, negative, or zero, indicating payments, rewards, or no transaction, respectively. We assume that users' utilities are quasi-linear; i.e., linear in the tax term. Therefore, the *total (security) cost* for a user i when she is assigned a tax t_i is given by:

$$\mathrm{g}_i(\mathbf{x}, t_i) := g_i(\mathbf{x}) + t_i \ , \tag{3}$$

where the tax t_i can in general be a function of the state of security $\mathbf{x}$.

In addition to implementing the socially optimal solution, incentive mechanisms are often designed so as to satisfy two properties. First, when using taxation, the mechanism designer prefers to maintain *weak budget balance (WBB)*; i.e., $\sum_{i=1}^{N} t_i \geq 0$. In contrast, $\sum_i t_i < 0$ implies a budget deficit, such that the implementation of the mechanism would require spending (a potentially large amount of) external resources by the designer.

In addition, it is desirable to design the mechanism in a way that users' *voluntary participation (VP)* conditions are satisfied; otherwise, the designer would need to enforce initial cooperation in the mechanism. Note the deliberate choice of the term voluntary participation as opposed to the commonly studied *individual rationality (IR)* constraint. Individual rationality commonly requires a user to prefer participation in a proposed mechanism to the outcome in which she receives no allocation of the good at all. In contrast, voluntary participation ensures that a user prefers implementing the socially optimal outcome while being assigned a tax t_i, to the outcome attained had she unilaterally opted out. Such distinction is important as security is a non-excludable public good; i.e., users can still benefit from the externalities generated by the participating users, even when opting out themselves. This is in contrast to games with excludable public goods, where voluntary participation and individual rationality are in fact equivalent.

Therefore, to enable the study of users' voluntary participation in the provision of non-excludable goods, we propose the concept of *exit equilibrium (EE)*. Consider a *loner* or *outlier*, who is unilaterally contemplating opting out of a proposed incentive mechanism. As the game considered here is one of full information, the remaining participating users, who are choosing a welfare maximizing solution for their $(N-1)$-user system, will have the ability to predict the best-response of the loner to their collective action, and thus choose their investments accordingly. As a result, the equilibrium investment profile when user i opts out is the Nash equilibrium of the game between the $N-1$ participating users and this loner. Formally, when user i is the outlier, the exit equilibrium $\hat{\mathbf{x}}^i$ is given by:

$$\begin{aligned} \hat{\mathbf{x}}^i_{-i} &= \arg\min_{\mathbf{x}_{-i}\succeq 0} \ \sum_{j\neq i} g_j(\mathbf{x}_{-i}, \hat{x}^i_i) \ , \\ \hat{x}^i_i &= \arg\min_{x_i \geq 0} \quad g_i(\hat{\mathbf{x}}^i_{-i}, x_i) \ . \end{aligned} \tag{4}$$

where $\mathbf{x}_{-i} := \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N\}$ denotes the profile of efforts of users other than i.

We note that the study of exit equilibria to understand users' unilateral deviations from socially optimal investment profiles is similar to the study of users' deviation from Nash equilibria: neither concept precludes the possibility that coalitions of deviating users can break the equilibrium. Nevertheless, as shown in Section III, there exists no incentive mechanism that can guarantee social optimality, weak budget balance, and voluntary participation, even against unilateral deviations, much less against higher order coalitions. We therefore only focus on unilateral exit strategies in this paper.

III. An Impossibility Result

In this section, we prove the following theorem:

Theorem 1: *There exists no tax-based incentive mechanism which can implement the socially optimal solution, while guaranteeing both weak budget balance and voluntary participation simultaneously, in all instances of security games.*

We prove this impossibility through two families of counter-examples. The first counter-example considers network structures with a star topology; the second family focuses on the particular class of weakest link risk functions.

A. Counter-example I: the star topology

Assume some tax-based incentive mechanism $\mathcal{M}$ is proposed for security games. Consider N users connected through the star topology with user 1 as the center, such that the security decisions of the center affects all leaves, but each leaf's investment only affects herself and the center. Formally, let the cost function of the center be given by:

$$g_1(\mathbf{x}) = f(x_1 + \sum_{j=2}^{N} x_j) + cx_1 \ ,$$

and that of all leaves $j \in \{2, \ldots, N\}$ be:

$$g_j(\mathbf{x}) = f(x_1 + x_j) + cx_j \ .$$

Here, $f(\cdot)$ is any function satisfying the assumptions in Section II-A. The investment cost functions $h_i(\cdot)$ are linear, with the same unit investment cost c for all users.

We first solve (2) to find the socially optimal investment profile $\mathbf{x}^*$. It is easy to see that for this graph, only the center will be investing in security, while all leaves free-ride on the resulting externality. This socially optimal investment profile $\mathbf{x}^*$ is given by:

$$\frac{\partial f}{\partial x}(x_1^*) = -\frac{c}{N}, \quad x_j^* = 0, \forall j = 2, \ldots, N \ .$$

Now, assume the center user is considering stepping out of the mechanism. To find the exit equilibrium profile $\hat{\mathbf{x}}^1$ resulting from this unilateral deviation, first note that the leaves' security decisions will not affect one another, so that the socially optimal investment profile for the $N-1$ leaves is the same as their myopic decisions. User 1 will also be choosing her individually optimal level of investment. Therefore, using (4), the exit equilibrium $\hat{\mathbf{x}}^1$ is:

$$\frac{\partial f}{\partial x}(\hat{x}_1^1) = -c, \quad \hat{x}_j^1 = 0, \forall j = 2, \ldots, N \ .$$

Finally, if any leaf user $j \in \{2, \ldots, N\}$ leaves the mechanism, the exit equilibrium $\hat{\mathbf{x}}^j$ will be given by:

$$\frac{\partial f}{\partial x}(\hat{x}_1^j) = -\frac{c}{N-1}, \quad \hat{x}_k^j = 0, \forall k = 2, \ldots, N \ .$$

We now use the socially optimal investment profile and the exit equilibria to evaluate voluntary participation and weak budget balance for mechanism $\mathcal{M}$. Assume $\mathcal{M}$ assigns a tax t_i^* to a participating user i. Then, voluntary participation will hold if and only if $\mathrm{g}_i(\mathbf{x}^*, t_i^*) \leq g_i(\hat{\mathbf{x}}^i), \ \forall i$, which reduces to:

$$\begin{aligned} t_1^* &\leq f(\hat{x}_1^1) - f(x_1^*) + c(\hat{x}_1^1 - x_1^*) \ , \\ t_j^* &\leq f(\hat{x}_1^j) - f(x_1^*), \ \forall j \in \{2, \ldots, N\} \ . \end{aligned}$$

The sum of these taxes is thus bounded by:

$$\begin{aligned} \sum_{i=1}^{N} t_i^* \leq & f(\hat{x}_1^1) - f(x_1^*) + c(\hat{x}_1^1 - x_1^*) \\ & + (N-1)(f(\hat{x}_1^j) - f(x_1^*)) \end{aligned}$$

However, the above sum can be negative, e.g., when $f(z) = \exp(-z)$ or $f(z) = \frac{1}{z}$, indicating that weak budget balance will fail regardless of how the taxes are determined in the mechanism $\mathcal{M}$.

Intuition: the failure of any mechanism $\mathcal{M}$ in guaranteeing SO, WBB, and VP in this topology, is due to the fact that the center node (a main investor) asks for a reward that the leaves (free-riders) are not willing to subsidize. Note that if the users were facing a choice between the center investing the socially optimal level, and staying at the Nash equilibrium, this problem would have not arisen, and it would be possible to guarantee all three properties. Nevertheless, as outlier leaf nodes can still enjoy a lower level of security subsidized by other participating leaves, their willingness to pay is limited, consequently not financing the reward requested by the center.

B. Counter-example II: weakest-link games

We again assume a general tax-based incentive mechanism $\mathcal{M}$ for security games. We focus on a family of security games which approximate the *weakest link* risk function $f_i(\mathbf{x}) = \exp(-\min_j x_j)$ [2], [5]. Intuitively, this model states that an attacker can compromise the security of an interconnected system by taking over the least protected node. To use this model in our current framework, we need a continuous, differentiable approximation of the minimum function. We use the approximation $\min_j x_j \approx -\frac{1}{\rho}\log\sum_j \exp(-\rho x_j)$, where the accuracy of the approximation is increasing in the constant $\rho > 0$. User i's cost function is thus given by:

$$g_i(\mathbf{x}) = (\sum_{j=1}^{N} \exp(-\rho x_j))^{1/\rho} + cx_i \ ,$$

where investment cost functions $h_i(\cdot)$ are assumed to be linear, with the same unit investment cost c for all users.

In this game, the socially optimal investment profile $\mathbf{x}^*$ is given by the solution to (2), which leads to:

$$N \exp(-\rho x_i^*)(\sum_{j=1}^{N} \exp(-\rho x_j^*))^{\frac{1}{\rho}-1} = c \ , \forall i.$$

By symmetry, all users will be exerting the same socially optimal level of effort:

$$x_i^* = \frac{1}{\rho} \ln \frac{N}{c^\rho} \ , \forall i \ .$$

Next, assume a user i unilaterally steps out of the mechanism, while the remaining users continue participating. The exit equilibrium profile $\hat{\mathbf{x}}^i$ can be determined using:

$$\begin{aligned} (N-1)\exp(-\rho \hat{x}_j^i)(\sum_{k \neq i} \exp(-\rho \hat{x}_k^i) + \exp(-\rho \hat{x}_i^i))^{\frac{1}{\rho}-1} = c \ , \\ \exp(-\rho \hat{x}_i^i)(\sum_{k \neq i} \exp(-\rho \hat{x}_k^i) + \exp(-\rho \hat{x}_i^i))^{\frac{1}{\rho}-1} = c \ . \end{aligned}$$

Solving the above, we get:

$$\begin{aligned} \hat{x}_i^i &= \frac{1}{\rho} \ln \frac{2^{1-\rho}}{c^\rho} \\ \hat{x}_j^i &= \frac{1}{\rho} \ln \frac{(N-1)2^{1-\rho}}{c^\rho} \ , \forall j \neq i \ . \end{aligned}$$

We now use the socially optimal investment profile and the exit equilibria to analyze users' participation incentives in a general mechanism $\mathcal{M}$, as well as the budget balance conditions. Denote by t_i^* the tax assigned to user i by $\mathcal{M}$.

The voluntary participation condition for a user i will hold if and only if $\mathrm{g}_i(\mathbf{x}^*, t_i^*) \leq g_i(\hat{\mathbf{x}}^i)$, which reduces to:

$$c(1+x_i^*) + t_i^* \leq c(2+\hat{x}_i^i) \Leftrightarrow t_i^* \leq c(1+\frac{1}{\rho}\ln\frac{2^{1-\rho}}{N}) \ . \quad (5)$$

On the other hand, for weak budget balance to hold, we need $\sum_i t_i^* \geq 0$. Nevertheless, by (5), we have:

$$\sum_i t_i^* \leq cN(1+\frac{1}{\rho}\ln\frac{2^{1-\rho}}{N}) \ .$$

It is easy to see that given ρ and for any $N > e^{\rho}2^{1-\rho}$, the above sum will always be negative, indicating a budget deficit for a general mechanism $\mathcal{M}$, regardless of how taxes are determined.

Intuition: here, the failure of any mechanism $\mathcal{M}$ occurs only when the number of players is large (given a finite ρ). This is because with a large number of participating users, the externality available to an outlying free-rider is high enough to dissuade her from participating. It is also interesting to point out that when $N \leq e^{\rho}2^{1-\rho}$, the Externality mechanism introduced in Section IV can guarantee SO, BB, and VP.

C. *A note on the nature of this impossibility result*

In Section V-A, we discuss in further detail the relation between this impossibility result and existing impossibility and possibility results in the literature. We point out that our impossibility result on a simultaneous guarantee of social optimality, voluntary participation, and weak budget balance, is demonstrated through two family of counter-examples. That is, we have shown that without prior knowledge of the graph structure or users' preferences, it is not possible for a designer to propose a *reliable* mechanism; that is, one which can promise to achieve SO, VP, and WBB, regardless of the realizations of utilities. Nevertheless, it may still be possible to design reliable mechanisms under a restricted problem space. With this in mind, we next analyze the class of weighted effort models, and aim to identify such positive instances, as well as the intuition behind the existence of each instance.

IV. Weighted Effort Games: Analysis and Simulations

In light of the impossibility result of Section III, in this section we set out to better understand the performance of existing incentive mechanisms in security games, and identify features of the problem environment that affect the properties attainable through given mechanisms. Specifically, we consider weighted effort games, and analyze the performance of the Pivotal and Externality mechanisms within this class.

A. *A Tale of Two Mechanisms*

Throughout this section, we will be studying the performance of two well-known tax-based incentive mechanisms, namely the Pivotal (VCG) and Externality mechanisms. We chose these mechanisms as they have been shown to simultaneously guarantee the achievement of social optimality, weak budget balance, and voluntary participation, in games of provision of *excludable* public goods. Our goal is hence to illustrate their inefficiencies in the provision of *non-excludable* public goods.

1) The Pivotal Mechanism: Groves mechanisms [9], [11], also commonly known as Vickery-Clarke-Groves (VCG) mechanisms, refer to a family of mechanisms in which, through the appropriate design of taxes for users with quasi-linear utilities, a mechanism designer can incentivize users to reveal their true preferences in dominant strategies, thus implementing the socially optimal solution. One particular instance of these mechanisms, the *Pivotal* mechanism, has been shown to further satisfy the participation constraints and achieve weak budget balance in many private and public good games [9], [12], [13]; however, this is not necessarily the case in security games. The taxes in the Pivotal mechanism for security games are given by:

$$t_i^P = \sum_{j\neq i} g_j(\mathbf{x}^*_{-i}, x_i^*) - \sum_{j\neq i} g_j(\hat{\mathbf{x}}^i_{-i}, \hat{x}^i_i) \ , \qquad (6)$$

where, $g_i(\mathbf{x})$ is user i's security cost function, $\mathbf{x}^* = (\mathbf{x}^*_{-i}, x_i^*)$ is the socially optimal solution, and $\hat{\mathbf{x}}^i = (\hat{\mathbf{x}}^i_{-i}, \hat{x}^i_i)$ is the exit equilibrium under user i's unilateral deviation. It can be shown that despite incentivizing users' voluntary participation and attaining the socially optimal solution, these taxes may generate a budget deficit for the designer in security games; we refer the interested reader to [14] for proofs and counterexamples.

2) The Externality Mechanism: we next introduce the Externality mechanism adapted from the work of Hurwicz in [15]. A main design goal of this mechanism is to guarantee a complete redistribution of taxes; i.e., strong budget balance. This mechanism has been adapted in [8], where it is shown to achieve social optimality, guarantee participation, and maintain a balanced budget, in allocation of power in cellular networks (i.e., an excludable public good). However, this is again not the case in security games. The tax terms t_i^E at the equilibrium of the Externality mechanism in security games are given by:

$$t_i^E(\mathbf{x}^*) = -\sum_{j=1}^{N} x_j^* \frac{\partial f_i}{\partial x_j}(x^*) - x_i^* \frac{\partial h_i}{\partial x_i}(x_i^*) \ . \qquad (7)$$

The interpretation is that by implementing this mechanism, each user i will be financing part of user $j \neq i$'s reimbursement. According to (7), this amount is proportional to the positive externality of j's investment on user i's utility. It can be shown that despite attaining the socially optimal solution and having a strongly balanced budget, these taxes may fail to satisfy users' voluntary participation constraints in security games; see [14] for proofs and counterexamples, and additional intuition behind the design of this mechanism.

B. *Choice of the risk function*

The appropriate choice of the risk functions $f_i(\cdot)$ for a given environment is based on factors such as the type of interconnection, the extent of interaction among users, and the type of attack. Several models of security interdependency have been proposed and studied in the literature; these include the *total effort*, *weakest link*, and *best shot* models considered in the seminal work of Varian [5], as well as the *weakest target* games studied in [16], the *effective investment* and *bad traffic* models in [17], and the *linear influence network* games in [18].

In this paper, we take the special case of *weighted effort* games, with exponential risks and linear investment cost functions. Formally, the total cost of a user i is given by:

$$\mathrm{g}_i(\mathbf{x}, t_i) = \exp(-\sum_{j=1}^{N} a_{ij}x_j) + cx_i + t_i \ . \tag{8}$$

Here, the coefficients $a_{ij} \geq 0$ determine the dependence of user i's risk on user j's action. We define the *dependence matrix* containing these coefficients as:

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} .$$

We isolate the effect of different features of the model on the performance of the two incentive mechanisms, by focusing on the following three sub-classes of this model:

1. Varying users' self-dependence:

$$A = \begin{pmatrix} a & 1 & \cdots & 1 \\ 1 & a & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & a \end{pmatrix} , \tag{9}$$

for both $a > 1$ and $a < 1$.

2. Effects of diversity, by breaking users into two groups of self-dependent and reliant users:

$$A = \begin{pmatrix} a_1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & a_1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & a_1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & \cdots & a_2 \end{pmatrix} , \tag{10}$$

for $a_1 > 1$ and $a_2 < 1$.

3. Making all users increasingly dependent on a single user:

$$A = \begin{pmatrix} a & 1 & \cdots & 1 \\ a & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ a & 1 & \cdots & 1 \end{pmatrix} , \tag{11}$$

for $a > 1$.

We present numerical results and intuitive interpretation for each of the above scenarios; formal analysis is given in the online appendix [10].

C. Effects of self-dependence

Consider a network of N users, with the dependence matrix given by (9), and total cost functions: [2]

$$\mathrm{g}_i(\mathbf{x}, t_i) = \exp(-ax_i - \sum_{j \neq i} x_j) + cx_i + t_i \ .$$

The following theorem characterizes the possible exit equilibria of this game under different parameter conditions, as

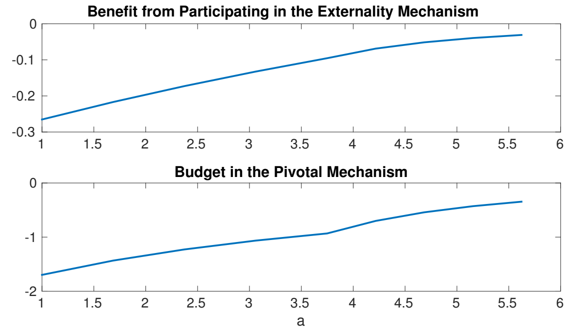

Fig. 1. Increasing self-dependence a

well as whether the voluntary participation conditions are satisfied under the Externality mechanism, and whether the Pivotal mechanism can operate without a budget deficit. The results are summarized in Table I.

Theorem 2: *For the security game described by the dependence matrix* (9)*:*

(i) *There exist five possible exit equilibria (summarized in Table I) depending on the values of the number of players N, self-dependence a, and cost of investment c.*

(ii) *Either of the Externality or Pivotal mechanisms can guarantee social optimality, voluntary participation, and weak budget balance, if and only if the realized exit equilibrium is ω or ζ of Table I.*

The proof, including formal derivations of the exit equilibria, as well as the analysis of the Pivotal and Externality mechanisms under each exit equilibrium, is presented in the online appendix [10].

Using simulations, we further examine the effect of changing a on the mechanisms' performance. In particular, we plot the sum of all taxes, $\sum_i t_i^P$, in the Pivotal mechanism. For the Externality mechanism, we plot $g_i(\hat{\mathbf{x}}^i) - \mathrm{g}_i(\mathbf{x}^*, t_i^E)$ per user i; i.e., the benefit of participation (in terms of cost reduction) for that user. We set $N = 6$ and $c = 1$. We then increase a, starting from $a = 1$, hence moving gradually from the exit equilibrium in [Case α] to [Case β]. Intuitively, by increasing users' self-dependence, a unit of investment becomes more effective for the user. As a result, we move towards an exit equilibrium in which outliers exert non-zero effort. Figure 1 illustrates the results. From our analysis and simulations, we make the following observations:

Higher self-dependence improves performance of mechanisms: from Fig. 1 we observe that (as predicted by the analysis) the Pivotal mechanism will always carry a deficit, while the Externality mechanism will always fail to guarantee voluntary participation. Nevertheless, as self-dependence increases, the performance of both mechanisms improves. This is because higher self-dependence (equivalently, lower interdependence) leads to closer to optimal investments by individual users in their exit equilibrium. Such users require smaller incentives to move to the optimal state, hence the

[2] We assume $c < a$, so as to ensure the existence of non-zero equilibria; i.e., at least one user exerts non-zero effort at any equilibrium of the game.

TABLE I

CAN SO, VP, AND WBB, HOLD SIMULTANEOUSLY? - EFFECT OF SELF-DEPENDENCE

	Exit Equilibrium	Parameter Conditions	VP in Externality	WBB in Pivotal
CASE α	$\hat{x}_i^i = 0, \hat{x}_j^i > 0$	$a > 1$, with N and c s.t. $(1 + \frac{N-2}{a})^{N-1} > (\frac{a}{c})^{a-1}$	Never	Never
CASE β	$\hat{x}_i^i > 0, \hat{x}_j^i > 0$	$a > 1$, with N and c s.t. $(1 + \frac{N-2}{a})^{N-1} < (\frac{a}{c})^{a-1}$	Never	Never
CASE γ	$\hat{x}_i^i = 0, \hat{x}_j^i > 0$	$a < 1$, any N and c	Never	Never
CASE ω	$\hat{x}_i^i > 0, \hat{x}_j^i = 0$	$a < 1$, with N and c s.t. $(1 + \frac{N-2}{a})^{a} < (\frac{a}{c})^{1-a}$	Always	Always
CASE ζ	$\hat{x}_i^i > 0, \hat{x}_j^i > 0$	$a < 1$, with N and c s.t. $(1 + \frac{N-2}{a})^{a} < (\frac{a}{c})^{1-a}$	Always	Always

reduced budget deficit of the Pivotal, and smaller participation costs in the Externality mechanism.

Coordinating on the least beneficial exit equilibrium for the outlier: from Table I, we also observe that if selection among multiple exit equilibria is possible, the Pivotal and Externality mechanism can simultaneously guarantee SO, VP, and WBB under the less beneficial ones. A less beneficial equilibrium can be one that requires a free-rider to become an investor when leaving the mechanism, or one that requires an investor to continue exerting effort when out (although possibly at a lower level). This can be seen by comparing Cases ω and ζ (in which outliers become the main investors or have to continue exerting effort when out, respectively) with Case γ (in which outliers become free-riders).

An exchange of favors: it is also interesting to highlight the nature of the positive instances of Cases ω and ζ of Table I: as users are mainly dependent on others' investments under these parameter conditions ($a < 1$), the incentive mechanisms can facilitate coordination among them, so that each increase their investments in return for improved investments by others.

D. Effects of diversity: self-dependent and reliant communities

Next, consider a collection of N users, with dependence matrix given by (10), and the following total cost functions:

$$\mathrm{g}_i(\mathbf{x}, t_i) = \exp(-a_{ii}x_i - \sum_{j \neq i} x_j) + cx_i + t_i \ .$$

We consider the two classes of self-dependent users N_1 for whom $a_{ii} = a_1, i \in N_1$, and the reliant users $N_2 = N - N_1$ for whom $a_{ii} = a_2, i \in N_2$. We let $c < a_2 < 1 < a_1$.

We formally show that at the socially optimal investment profile, self-dependent users in N_1 will be main investors, while reliant users in N_2 are free-riders (see online appendix [10]). We further show that outliers, both from N_1 or N_2, may either become free-riders or have to exert effort at their exit equilibria, depending on the specific problem parameters. However, we only provide a partial characterization of the socially optimal and exit equilibrium for this subclass, as the equations determining these profiles do not have a closed form solution.

We therefore use numerical simulations to illustrate the N_1 and N_2 users' benefits (in terms of cost reduction) from participating in the Externality mechanism, as well as the budget of the Pivotal mechanism. In particular, we set $N = 10$, $N_1 = 8$, $c = 0.05$, $a_2 = 0.1$, and change $a_1 \in [1, 10]$; see Fig. 2. Under these parameter conditions, an outlier user from N_2 will become the main investor in her exit equilibrium, while all remaining users free-ride. For users in N_1, as their self-dependence a_1 increases, we move from an exit equilibrium in which they are free-riders, to one in which they continue exerting (lower) effort even when opting out.

When exit equilibrium for both classes are less beneficial, all participation incentives in the Externality mechanism are satisfied: Fig. 2 illustrates how users' voluntary participation constraints in the Externality mechanism are highly affected by their actions in the exit equilibria. Here, the VP conditions for users in N_2 are always satisfied, as they would exert effort at their exit equilibrium. For users in N_1 however, VP conditions are satisfied only after their exit equilibrium switches to one where they exert effort.

Transfer of resources between the two communities: we further observe that the Pivotal mechanism carries a budget surplus for all values of a_1, given that a_2 is relatively small. This is because the highly reliant N_2 users are willing to finance the increased effort of N_1 users.

E. Effects of a dominant user

Consider a collection of N users, with dependence matrix given by (11), and total cost functions:

$$\mathrm{g}_i(\mathbf{x}, t_i) = \exp(-ax_1 - \sum_{j=2}^{N} x_j) + cx_i + t_i \ ,$$

where $c < 1 < a$, and user 1 is the dominant user. We show that in a socially optimal profile, as well as for exit equilibria of non-dominant users, only user 1 will be exerting effort. When the dominant user opts out of the mechanism, however, she may become a main investor or free-rider.

The following theorem characterizes the possible exit equilibria and parameter conditions for which each is possible, as well as the performance of both mechanisms. The results are summarized in Table II.

Theorem 3: *For the security game described by the dependence matrix* (11)*:*

(i) *There exist two possible exit equilibria (summarized in Table II) depending on the values of the number of players N and dependence on the dominant user a.*

(ii) *None of the Externality or Pivotal mechanisms can guarantee social optimality, voluntary participation, and weak budget balance, regardless of the exit equilibrium.*

The proof is presented in the online appendix [10].

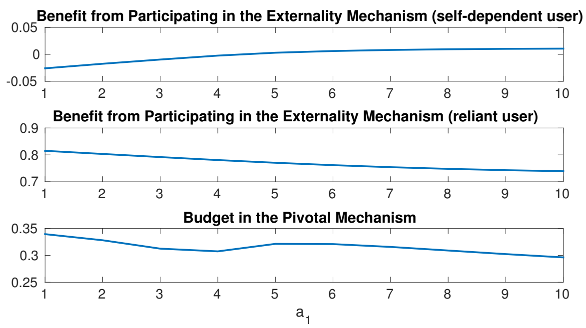

Fig. 2. Two diverse communities of self-dependent and reliant users

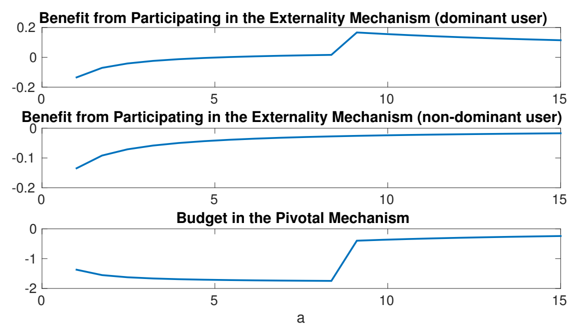

Fig. 3. Main investors may opt out: effects of a single dominant user

TABLE II
CAN SO, VP, AND WBB, HOLD SIMULTANEOUSLY? - SINGLE DOMINANT USER

	Exit Equilibrium	Parameter Conditions	VP in Externality	WBB in Pivotal
CASE α	$\hat{x}_1^1 = 0, \hat{x}_j^1 > 0, \forall j \neq 1, \hat{x}_1^i > 0, \hat{x}_j^i = 0, \forall i, j \neq 1$	$a < N - 1$	Never	Never
CASE β	$\hat{x}_1^1 > 0, \hat{x}_j^1 = 0, \forall j \neq 1, \hat{x}_1^i > 0, \hat{x}_j^i = 0, \forall i, j \neq 1$	$a > N - 1$	Never	Never

Using simulations, we further illustrate the effect of increasing a, the dependence on the dominant user, on users' benefits from participating in the Externality mechanism (i.e., $g_i(\hat{\mathbf{x}}^i) - g_i(\mathbf{x}, t_i^E)$), as well as the budget of the Pivotal mechanism (i.e., $\sum_i t_i^P$). We set $N = 10$, $c = 0.45$, and $a \in [1, 15]$. As a increases, the dominant user's exit equilibrium switches from free-riding to investing when opting out.

Either main investors or free-riders may opt out: as illustrated in Fig. 3, the VP conditions of free-riders in the Externality mechanism are never satisfied: these users can avoid paying taxes to the dominant user, while others pay her to increase her investment. More interesting is the fact that the VP conditions for the main investor may also fail to hold. This is because when user 1's exit equilibrium does not require her to exert effort, and the externality generated by her is small (i.e.; small a), the collected taxes are not enough to persuade this dominant user to increase her effort level. Furthermore, we observe that although the Pivotal mechanism needs to give out a smaller reward to the dominant user for larger a (hence the jump in the third plot in Fig. 3), it still fails to avoid a deficit due to the small willingness of free-riders to pay the taxes required to cover this reward.

V. RELATED WORK

A. Existing possibility and impossibility results

The presented impossibility result is different from those in the existing literature, in either the selected equilibrium solution concept, the set of properties the mechanism is required to satisfy, or the space of utility functions. For example, the Myerson and Satterthwaite result [9] differs from our work in both solution concept (Bayesian Nash vs. full Nash implementation in our work) and imposing the stronger requirement of *strong* budget balance.

The most closely related impossibility result to our work is that of [19], which also studies impossibility results in the provision of non-excludable public goods. Our adoption of the term voluntary participation as opposed to individual rationality is similar to this work. However, our paper differs from [19] in that the latter studies Cobb-Douglas utilities, whereas we consider quasi-linear utilities. More importantly, outliers in [19] can only exert zero effort (as the model considered is production of a single good with constant return to scale technology), whereas outliers in our model can best-respond to the collective action of the participating users, potentially exerting non-zero effort and contributing to the provision of the good even when opting out. The notion of exit equilibrium is introduced to fully capture this distinction.

The current work should also be viewed in conjunction with existing possibility results, notably [8], [9], [12], [13]. These papers show that under the exact same utility functions and informational constraints, had users' outside options been zero, the Externality and Pivotal mechanisms would simultaneously guarantee social optimality, voluntary participation, and weak budget balance. Therefore, the goal of our work is not solely to prove the impossibility of the design, but to highlight the important distinction users' outside options make in the choice of a mechanism.

B. Public good provision games

The problem of incentivizing optimal security investments in an interconnected system is one example of problems concerning the provision of non-excludable public goods in social and economic networks. Other examples include creation of new parks or libraries at neighborhood level in cities [20], reducing pollution by neighboring towns [21], or spread of innovation and research in industry [22]. We summarize some of the work most relevant to the current paper.

The authors in [22] introduce a network model of public goods, and study different features of its Nash equilibria. This model is equivalent to a total effort game with linear investment costs and a general interdependence graph. The work in [20] studies existence, uniqueness, and closed form of the Nash equilibrium, in a class of games for which best-responses are linear in other players' actions. The aforementioned work differ from the current paper in that they focus only on the Nash equilibrium of the games, whereas we study the mechanism design problem, therefore analyzing socially optimal investments and exit equilibria.

The work of [21] is also relevant to our work, as it studies Pareto efficient outcomes in the provision of non-excludable public goods, and establishes a connection between efforts at a Lindahl outcome and the eigenvalue centrality vector of a suitably defined benefits matrix. Our work in this paper is on the study of voluntary participation in such environments, which [21] also mentions as a direction of future work.

Finally, in the context of security games, our work is most related to [17], [18]. The weighted effort risk model is a generalization of the total effort model in [5], and is similar to the effective investment model in [17] and the linear influence network game in [18]. The linear influence models in [18] have been proposed to study properties of the interdependence matrix affecting the existence and uniqueness of the Nash equilibrium. The effective investment model in [17] has been considered to determine a bound on the price of anarchy gap, i.e. the gap between the socially optimal and Nash equilibrium investments, in security games. Our work on the above model fills a gap within this literature as well, by (1) introducing the study of exit equilibria, (2) analyzing the general mechanism design problem, and (3) considering the effect of users' interdependence on the performance of incentive mechanisms.

VI. Conclusion

We introduced the notion of exit equilibrium to study voluntary participation of users in mechanisms for provision of non-excludable public goods, such as security. Our proposed equilibrium concept accounts for the spillovers available to users opting out of the mechanism, as well as their possible contribution to the public good (here, the state of security) even when opting out. We have shown the fundamental result that, with these outside options, it is not possible to design a tax-based incentive mechanism to implement the socially optimal solution while guaranteeing voluntary participation and maintaining a weakly balanced budget, without additional information on the graph structure and users' preferences. We have further identified several specific problem environments in which this conflict does not emerge. These include environments that allow for a transfer of funds across two diverse communities of users, an exchange of favors among highly interdependent entities, or the coordination of participating users on the least beneficial exit equilibrium to the outlier.

An implication of our result is that, when a designer lacks additional information about the specifics of the environment, and is interested in guaranteeing voluntary participation without spending additional external resources, she may opt to forgo social optimality, instead reliably achieving a sub-optimal state of security. Characterizing the best attainable sub-optimal solution, as well as mechanisms leading to it, is a main direction of future work.

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APPENDIX

EXIT EQUILIBRIA OF THE WEIGHTED EFFORT MODEL - VARYING SELF-DEPENDENCE

In this appendix, we solve for the socially optimal investment profile, and identify the possible exit equilibria, and parameter conditions under which each equilibrium is possible.

The socially optimal investment profile in this game will be given by:

$$x_i^* = \frac{1}{a+N-1} \ln \frac{a+N-1}{c}, \forall i \ .$$

To find the exit equilibrium when a user i steps out, $\hat{\mathbf{x}}^i$, we can write the first order conditions on the users' cost minimization problems. To simplify notation, we denote $x := \hat{x}_i^i$ and $y := \hat{x}_j^i, \forall j \neq i$. The system of equation determining x and y is given by:

$$\begin{aligned} -a\exp(-ax-(N-1)y)+c &\geq 0 \\ -(a+N-2)\exp(-x-(a+N-2)y)+c &\geq 0 \ . \end{aligned} \quad (12)$$

There are four possible exit equilibria, depending on the whether x and/or y are non-zero. We look at each case separately.

Exit equilibria with $x>0, y>0$

Intuitively, when user i steps out, both sides continue to invest in security, perhaps at reduced levels, but no user is fully free-riding. We would need the following to hold simultaneously:

$$\begin{aligned} -a\exp(-ax-(N-1)y)+c &= 0 \\ -(a+N-2)\exp(-x-(a+N-2)y)+c &= 0 \ . \end{aligned}$$

Let $L_1 = \log\frac{a}{c}$ and $L_2 = \log\frac{a+N-2}{c}$. Solving for x, y leads to:

$$\begin{aligned} x &= \frac{1}{(a-1)(a+N-1)} \log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^{-(N-1)} \\ y &= \frac{1}{(a-1)(a+N-1)} \log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^{a} \ . \end{aligned}$$

To find the range of parameters for which the above holds, we need to ensure that x, y are indeed positive.

- If $a>1$, then $y>0$. For $x>0$, we need:
$$(\frac{a}{c})^{a-1} > (1+\frac{N-2}{a})^{N-1}$$
- If $a<1$, then $x>0$. For $y>0$, we need:
$$(1+\frac{N-2}{a})^{a} < (\frac{a}{c})^{1-a}$$

Exit equilibria with $x>0, y=0$

In this case, the participating users revert to investing zero, so that the outlier is forced to increase its investment:

$$\begin{aligned} -a\exp(-ax)+c &= 0 \\ -(a+N-2)\exp(-x)+c &> 0 \ . \end{aligned}$$

As a result, we get $x=\frac{1}{a}\log\frac{a}{c}$. For this to be consistent with the second condition, we require:

$$(1+\frac{N-2}{a})^{a} < (\frac{a}{c})^{1-a}$$

The above always fails to hold for $a>1$, as the LHS is always more than 1, while the RHS is surely less than 1 by the assumption $a>c$. Intuitively, when self-dependence is higher than co-dependence on the outlier, the remaining users will not rely solely on externalities and continue investing when user i steps out.

For $a<1$ on the other hand, for a small enough c (which in turn leads to higher investment x be the outlier), the equation may hold.

Exit equilibria with $x=0, y>0$

This means that the loner free-rides, so that we have:

$$\begin{aligned} -a\exp(-(N-1)y)+c &> 0 \\ -(a+N-2)\exp(-(a+N-2)y)+c &= 0 \ . \end{aligned}$$

As a result, we get $y=\frac{1}{a+N-2}\log\frac{a+N-2}{c}$. For this to be consistent with the first condition, we need:

$$(1+\frac{N-2}{a})^{N-1} > (\frac{a}{c})^{a-1}$$

Note that this always hold for $a<1$, but not necessarily for $a>1$.

Exit equilibria with $x=0, y=0$

We would need the following to hold simultaneously:

$$\begin{aligned} -a+c &> 0 \\ -(a+N-2)+c &> 0 \ , \end{aligned}$$

which will never hold, as we initially required that $c<a$.

BB AND VP IN EXIT EQUILIBRIA - VARYING SELF-DEPENDENCE

In this appendix, we separately analyze each of the possible cases identified in Appendix VI, summarized in Table I. Specifically, we are interested in the Budget balance condition under the Pivotal mechanism, and users' participation incentives in the Externality mechanism.

Case α: fails BB, fails VP

In this case, the underlying parameters satisfy $a>1$ and $(1+\frac{N-2}{a})^{N-1} > (\frac{a}{c})^{a-1}$. As a result, the exit equilibrium (EE) is such that $x=0$, and $y=\frac{1}{a+N-2}\log\frac{a+N-2}{c}$. Therefore, the costs of users at the SO and EE are given by:

$$\begin{aligned} g_j(\mathbf{x}^*) &= \frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}), \forall j \\ g_j(\hat{\mathbf{x}}^i) &= \frac{c}{a+N-2}(1+\log\frac{a+N-2}{c}), \forall j \neq i \\ g_i(\hat{\mathbf{x}}^i) &= \frac{c}{a+N-2}^{\frac{N-1}{a+N-2}} \end{aligned}$$

a) Budget Balance in the Pivotal mechanism: Note that $\frac{1+\log z}{z}$ is a decreasing function of z. Thus, $g_j(\hat{\mathbf{x}}^i) > g_j(\mathbf{x}^*)$ for all j, resulting in $t_i^P < 0$, indicating rewards to all users i, and thus a budget deficit in all scenarios. Intuitively, although when a user i steps out, other users have to invest less in security, thus decreasing their direct investment costs, still their overall security costs go up as a result of the increased risks. Consequently, each user i should be payed a reward to be kept in the mechanism, resulting in a budget deficit.

b) Voluntary Participation in the Externality mechanism: Voluntary participation will hold if and only if $g_i(\hat{\mathbf{x}}^i) \geq g_i(\mathbf{x}^*)$, that is:

$$\begin{aligned}
&\frac{c}{a+N-2}^{\frac{N-1}{a+N-2}} \geq \frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}) \\
\Leftrightarrow &\frac{c}{a+N-2}^{N-1} \\
&\geq (\frac{c}{a+N-1})^{a-1+N-1}(1+\log\frac{a+N-1}{c})^{a-1+N-1} \\
\Leftrightarrow &(\frac{a+N-1}{a+N-2})^{N-1}(1+\frac{N-1}{a})^{a-1}(\frac{a}{c})^{a-1} \\
&-(1+\log\frac{a}{c}(1+\frac{N-1}{a}))^{a+N-2} \geq 0
\end{aligned}$$

Based on the last inequality, define the function $g(z) := \kappa_1 z^{a-1} - (1+\log z)^{a+N-2}$. This function is increasing in z. As a result, it obtains its maximum when z reaches its maximum value, which by the initial condition is given by $\frac{a}{c} = (1+\frac{N-2}{a})^{\frac{N-1}{a-1}}$. Thus,

$$\begin{aligned}
g_{max} = &(\frac{a+N-1}{a+N-2})^{N-1}(1+\frac{N-1}{a})^{a-1}(1+\frac{N-2}{a})^{N-1} \\
&-(1+\log(1+\frac{N-2}{a})^{\frac{N-1}{a-1}}(1+\frac{N-1}{a}))^{a+N-2} \\
\leq &(1+\frac{N-1}{a})^{a+N-2} - (1+\log(1+\frac{N-1}{a}) \\
&+\frac{N-1}{a-1}\log(1+\frac{N-2}{a}))^{a+N-2} \\
\leq &(1+\frac{N-1}{a})^{a+N-2} - (1+\log(1+\frac{N-1}{a}) \\
&+\frac{N-1}{a}\log(1+\frac{N-2}{a}))^{a+N-2}
\end{aligned}$$

Let $z := \frac{N-1}{a}$, and define $f(z) := \log(1+z) + z\log(1+z-\frac{1}{a}) - z$ (i.e., we are assuming a fixed a). The derivative of this function wrt z is given by:

$$\frac{1}{1+z} + \log(1+z-\frac{1}{a}) + \frac{z}{1+z-\frac{1}{a}} - 1 =$$

$$\log(1+z-\frac{1}{a}) + \frac{\frac{1}{a}z}{(1+z)(1-\frac{1}{a}+z)} \; .$$

As the above is positive for all $a > 1$, we conclude that $f(z)$ is an increasing function in z. Furthermore, $\lim_{z\to 0} f(z) = 0$, which in turn means that $f(z) \geq 0, \forall z \geq 0$, and therefore, g_{max} is always non-positive. This in turn means that the VP condition can never be satisfied.

Case β: fails BB, fails VP

For this case, the underlying parameters satisfy $a > 1$ and $(1+\frac{N-2}{a})^{N-1} < (\frac{a}{c})^{a-1}$. As a result, the exit equilibrium (EE) is such that $x > 0, y > 0$, and are given by $x = \frac{1}{(a-1)(a+N-1)}\log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^{-(N-1)}$ and $y = \frac{1}{(a-1)(a+N-1)}\log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^{a}$. Therefore, the costs of users at the SO and EE are given by:

$$\begin{aligned}
g_j(\mathbf{x}^*) &= \frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}), \forall j \\
g_j(\hat{\mathbf{x}}^i) &= \frac{c}{a+N-2} \\
&+ \frac{c}{(a-1)(a+N-1)}\log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^{a}, \forall j \neq i \\
g_i(\hat{\mathbf{x}}^i) &= \frac{c}{a} \\
&+ \frac{c}{(a-1)(a+N-1)}\log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^{-(N-1)}
\end{aligned}$$

c) Budget Balance in the Pivotal mechanism: For the mechanism to have a budget deficit we would need $g_j(\hat{\mathbf{x}}^i) \geq g_j(\mathbf{x}^*)$, which holds if and only if:

$$\begin{aligned}
&\frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}) \leq \\
&\quad \frac{c}{a+N-2} + \frac{c}{(a-1)(a+N-1)}\log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^{a} \\
\Leftrightarrow &1+\log\frac{a}{c}(1+\frac{N-1}{a}) \leq \\
&\quad 1+\frac{1}{a+N-2}+\log\frac{a}{c}(1+\frac{N-2}{a})^{\frac{a}{a-1}} \\
\Leftrightarrow &\log(1+\frac{N-1}{a}) \leq \frac{1}{a+N-2} + \frac{a}{a-1}\log(1+\frac{N-2}{a}) \\
\Leftarrow &\log(1+\frac{N-1}{a}) \leq \frac{1}{a+N-2} + \log(1+\frac{N-2}{a}) \\
\Leftrightarrow &\log(1+\frac{1}{a+N-2}) \leq \frac{1}{a+N-2}
\end{aligned}$$

The last line is true because $\log(1+x) \leq x$, for all $x > 0$. Therefore, the mechanism always carries a budget deficit.

d) Voluntary Participation in the Externality mechanism: The mechanism fails voluntary participation if and only if:

$$\begin{aligned}
&\frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}) \geq \\
&\quad \frac{c}{a} + \frac{c}{(a-1)(a+N-1)}\log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^{-(N-1)} \\
\Leftrightarrow &1+\log\frac{a}{c}(1+\frac{N-1}{a}) \geq \\
&\quad 1+\frac{N-1}{a}+\log\frac{a}{c}(1+\frac{N-2}{a})^{\frac{-(N-1)}{a-1}} \\
\Leftrightarrow &\log(1+\frac{N-1}{a}) + \frac{N-1}{a-1}\log(1+\frac{N-2}{a}) \geq \frac{N-1}{a} \\
\Leftarrow &\log(1+\frac{N-1}{a}) + \frac{N-1}{a}\log(1+\frac{N-1}{a}-\frac{1}{a}) \geq \frac{N-1}{a}
\end{aligned}$$

Let $z := \frac{N-1}{a}$, and define $f(z) := \log(1+z) + z\log(1+z-\frac{1}{a}) - z$ (i.e., we are assuming a fixed a). The derivative of this function wrt z is given by:

$$\frac{1}{1+z} + \log(1+z-\frac{1}{a}) + \frac{z}{1+z-\frac{1}{a}} - 1 =$$

$$\log(1+z-\frac{1}{a}) + \frac{\frac{1}{a}z}{(1+z)(1-\frac{1}{a}+z)} .$$

As the above is positive for all $a > 1$, we conclude that $f(z)$ is an increasing function in z. Furthermore, $\lim_{z\to 0} f(z) = 0$, which in turn means that $f(z) \geq 0, \forall z \geq 0$, and therefore, that the VP condition always fails to hold under these parameter settings.

Case γ: fails BB, fails VP

Here, we only require that $a < 1$, and all other values of N or c will guarantee the existence of an equilibrium $x = 0$ and $y = \frac{1}{a+N-2}\log\frac{a+N-2}{c}$. This is thus parallel with Case α. Users' costs in the SO and EE are similarly given by:

$$\begin{aligned}
g_j(\mathbf{x}^*) &= \frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}), \forall j\\
g_j(\hat{\mathbf{x}}^i) &= \frac{c}{a+N-2}(1+\log\frac{a+N-2}{c}), \forall j \neq i\\
g_i(\hat{\mathbf{x}}^i) &= \frac{c}{a+N-2}^{\frac{N-1}{a+N-2}}
\end{aligned}$$

e) Budget Balance in the Pivotal mechanism: Note that $\frac{1+\log z}{z}$ is a decreasing function of z. Thus, $g_j(\hat{\mathbf{x}}^i) > g_j(\mathbf{x}^*)$ for all j, resulting in $t_i^P < 0$, indicating rewards to all users i, and thus a budget deficit in all scenarios (exactly similar to case α).

f) Voluntary Participation in the Externality mechanism: Voluntary participation will fail if and only if $g_i(\hat{\mathbf{x}}^i) \leq g_i(\mathbf{x}^*)$, that is:

$$\begin{aligned}
&\frac{c}{a+N-2}^{\frac{N-1}{a+N-2}} \leq \frac{c}{a+N-1}(1+\log\frac{a+N-1}{c})\\
\Leftrightarrow &\frac{c}{a+N-2}^{N-1} \leq\\
&\quad(\frac{c}{a+N-1})^{a-1+N-1}(1+\log\frac{a+N-1}{c})^{a-1+N-1}\\
\Leftrightarrow &(\frac{\frac{a}{c}(1+\frac{N-1}{a})}{1+\log\frac{a}{c}(1+\frac{N-1}{a})})^{a-1} \geq (\frac{1+\log\frac{a}{c}(1+\frac{N-1}{a})}{1+\frac{1}{a+N-2}})^{N-1}
\end{aligned}$$

First, we note that the RHS is always greater than 1, as $1+\log x \leq x$. On the other hand, since $a < 1$, $\frac{1}{a+N-2} < \log\frac{a}{c}(1+\frac{N-1}{a})$ holds for all $N \geq 3$, so that the LHS will be less than 1. Therefore, the VP condition always fails.

Case ζ: has BB, has VP

This case has equilibrium investments similar to case β, but under parameter conditions $a < 1$, and $(1+\frac{N-2}{a})^a < (\frac{a}{c})^{1-a}$. Therefore, we have the following costs for the users:

$$\begin{aligned}
g_j(\mathbf{x}^*) &= \frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}), \forall j\\
g_j(\hat{\mathbf{x}}^i) &= \frac{c}{a+N-2}\\
&\quad+\frac{c}{(a-1)(a+N-1)}\log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^a, \forall j \neq i\\
g_i(\hat{\mathbf{x}}^i) &= \frac{c}{a}\\
&\quad+\frac{c}{(a-1)(a+N-1)}\log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^{-(N-1)}
\end{aligned}$$

g) Budget Balance in the Pivotal mechanism: For the mechanism to have budget balance we would need $g_j(\hat{\mathbf{x}}^i) \leq g_j(\mathbf{x}^*)$, which holds if and only if:

$$\begin{aligned}
&\frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}) \geq\\
&\quad\frac{c}{a+N-2} + \frac{c}{(a-1)(a+N-1)}\log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^a\\
\Leftrightarrow &1+\log\frac{a}{c}(1+\frac{N-1}{a}) \geq\\
&\quad 1+\frac{1}{a+N-2}+\log\frac{a}{c}(1+\frac{N-2}{a})^{\frac{a}{a-1}}\\
\Leftrightarrow &\log(1+\frac{N-1}{a}) \geq \frac{1}{a+N-2} + \frac{a}{a-1}\log(1+\frac{N-2}{a})\\
\Leftarrow &\log(1+\frac{N-1}{a}) \geq \frac{1}{a+N-2}
\end{aligned}$$

The last line follows from the previous because $a < 1$, and is true because its LHS is $\geq \log N$ and its RHS is $\leq 1/(N-1)$. Therefore, the mechanism always has budget balance in this scenario.

h) Voluntary Participation in the Externality mechanism: The mechanism has voluntary participation if and only if:

$$\begin{aligned}
&\frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}) \leq\\
&\quad\frac{c}{a} + \frac{c}{(a-1)(a+N-1)}\log(\frac{a}{c})^{a-1}(1+\frac{N-2}{a})^{-(N-1)}\\
\Leftrightarrow &1+\log\frac{a}{c}(1+\frac{N-1}{a}) \leq\\
&\quad 1+\frac{N-1}{a}+\log\frac{a}{c}(1+\frac{N-2}{a})^{\frac{-(N-1)}{a-1}}\\
\Leftrightarrow &\log(1+\frac{N-1}{a})(1+\frac{N-2}{a})^{\frac{N-1}{a-1}} \leq \frac{N-1}{a}
\end{aligned}$$

The last statement holds because the second element in the logarithm is always less than 1, due to $a < 1$, and the result follows as $\log(1+z) \leq z$, for all $z > 0$.

Case ω: has BB, has VP

The last case in realized under parameter settings $a < 1$ and $(1+\frac{N-2}{a})^a < (\frac{a}{c})^{1-a}$, and $x = \log\frac{a}{c}$ and $y = 0$ is the possible exit equilibrium. The users' costs in the SO and EE here are given by:

$$\begin{aligned}
g_j(\mathbf{x}^*) &= \frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}), \forall j\\
g_j(\hat{\mathbf{x}}^i) &= (\frac{c}{a})^{\frac{1}{a}}, \forall j \neq i\\
g_i(\hat{\mathbf{x}}^{\mathbf{i}}) &= \frac{c}{a}(1+\log\frac{a}{c}) \ .
\end{aligned}$$

i) Budget Balance in the Pivotal mechanism: First we use $(1+\frac{N-2}{a})^a < (\frac{a}{c})^{1-a}$ to conclude that $(\frac{c}{a})^{\frac{1}{a}} \leq \frac{c}{a+N-2}$. Now, for the mechanism to have budget balance we would need $g_j(\hat{\mathbf{x}}^i) \leq g_j(\mathbf{x}^*)$, which holds if and only if:

$$\begin{aligned}
&\frac{c}{a+N-1}(1+\log\frac{a+N-1}{c}) \geq (\frac{c}{a})^{\frac{1}{a}}\\
\Leftarrow & 1+\log\frac{a}{c}(1+\frac{N-1}{a}) \geq 1+\frac{1}{a+N-2}\\
\Leftarrow & \log(1+\frac{N-1}{a}) \geq \frac{1}{a+N-2}
\end{aligned}$$

where the last line line follows from the previous because $\frac{a}{c} > 1$, and is true because its LHS is $\geq \log N$ and its RHS is $\leq 1/(N-1)$. Therefore, the mechanism always has budget balance in this scenario.

j) Voluntary Participation in the Externality mechanism: As $\frac{1+\log x}{x}$ is a decreasing function in x when $x > 1$, and $1 < \frac{a}{c} < \frac{a+N-1}{c}$, the costs when staying out are higher for user i. Therefore VP is satisfied in the Externality mechanism in this case.

Exit equilibria of the weighted effort model - two classes of self-dependence

In this appendix, we present a (partial) analysis of possible exit equilibria, and parameter conditions under which each is possible, for the wighted effort family described in Section IV-D. Denoting the investments of the users in N_1 and N_2 by x_1 and x_2, respectively, the socially optimal investment profile in this game is determined according to the first order conditions on users' cost minimization problems:

$$\begin{aligned}
&-(a_1+N_1-1)\exp(-(a_1+N_1-1)x_1-N_2x_2)\\
&\quad -N_2\exp(-N_1x_1-(a_2+N_2-1)x_2)+c \geq 0 \ ,\\
&-N_1\exp(-(a_1+N_1-1)x_1-N_2x_2)\\
&\quad -(a_2+N_2-1)\exp(-N_1x_1-(a_2+N_2-1)x_2)+c \geq 0 \ .
\end{aligned}$$

It is easy to see that at this socially optimal solution, $x_2 = 0$, and x_1 is given by:

$$\begin{aligned}
&(a_1+N_1-1)\exp(-(a_1+N_1-1)x_1)\\
&\quad +N_2\exp(-N_1x_1) = c \ .
\end{aligned}$$

In general, the above equation does not have a closed form solution. However, it is possible to find a lower bound and an upper bound on the solution x_1^*. It is also possible to solve for x_1^* numerically. Once we solve for this socially optimal investment, we can determine the taxes assigned by the Externality mechanism:

$$\begin{aligned}
t_i^E &= (a_1+N_1-1)x_1\exp(-(a_1+N_1-1)x_1)-cx_1,\\
&\qquad \forall i \in N_1 \ ,\\
t_i^E &= N_1x_1\exp(-N_1x_1), \forall i \in N_2 \ .
\end{aligned}$$

Note that the sum of the above taxes is indeed zero. Also, it is interesting to note that as expected, the free-riders in N_2 always pay a tax, while the main investors in N_1 receive a subsidy. In order to find exit equilibria in this family of games, we would again need to solve equations with a similar format to that of the socially optimal solution, which in general lack a closed form solution. We therefore do not include a full analysis of this scenario, and limit our discussion to some interesting features of the possible exit equilibria.

Exit equilibrium: a user from N_1 leaving

Let x denote the investment of the deviating user from group N_1, and y_1 and y_2 denote the investments of users remaining in N_1 and N_2. We have the following system of equations:

$$\begin{aligned}
&-a_1\exp(-a_1x-(N_1-1)y_1-N_2y_2)+c\\
&\qquad \geq 0 \ ,\\
&-(a_1+N_1-2)\exp(-x-(a_1+N_1-2)y_1-N_2y_2)\\
&\qquad -N_2\exp(-x-(N_1-1)y_1-(a_2+N_2-1)y_2)+c\\
&\qquad \geq 0 \ ,\\
&-(N_1-1)\exp(-x-(a_1+N_1-2)y_1-N_2y_2)\\
&-(a_2+N_2-1)\exp(-x-(N_1-1)y_1-(a_2+N_2-1)y_2)+c\\
&\qquad \geq 0 \ .
\end{aligned}$$

By an argument similar to that in the derivation of the SO solution in IV-D, we will always have $y_2 = 0$. As a result, the system of equations reduces to:

$$\begin{aligned}
&-a_1\exp(-a_1x-(N_1-1)y_1)+ && c \geq 0 \ ,\\
&-(a_1+N_1-2)\exp(-x-(a_1+N_1-2)y_1) && \\
&\quad -N_2\exp(-x-(N_1-1)y_1)+ && c \geq 0 \ .
\end{aligned}$$

We consider the following possible cases, depending on whether x and/or y_1 are non-zero.

Exit equilibria with $x > 0, y_1 > 0$: This would require a solution to the following system of equations:

$$\begin{aligned}
&a_1\exp(-a_1x-(N_1-1)y_1) && = c \ ,\\
&(a_1+N_1-2)\exp(-x-(a_1+N_1-2)y_1) && \\
&\quad +N_2\exp(-x-(N_1-1)y_1) && = c \ .
\end{aligned}$$

The above equations tell us that:

$$a_1x+(N_1-1)y_1 = \log\frac{a_1}{c} \ .$$

We can now substitute y_1 in the second equation to find x:

$$\begin{aligned}
&(a_1+N_1-2)\exp(-[\log\frac{a_1}{c}-(a_1-1)x]\\
&\quad -\frac{a_1-1}{N_1-1}[\log\frac{a_1}{c}-a_1x])\\
&\quad +N_2\exp(-[\log\frac{a_1}{c}-(a_1-1)x]) = c \ ,
\end{aligned}$$

which can be solved numerically.

Exit equilibria with $x = 0, y_1 > 0$: This would require:

$$\begin{aligned} &a_1 \exp(-(N_1-1)y_1) && \le c\ , \\ &(a_1+N_1-2)\exp(-(a_1+N_1-2)y_1) && \\ &\qquad + N_2 \exp(-(N_1-1)y_1) && = c\ . \end{aligned}$$

The first equation above can be used to find a lower-bound on y_1. Intuitively, the investments y_1 should be high enough for the outlier to decide against investing (i.e., set $x = 0$). We have:

$$y_1 \ge \frac{1}{N_1-1}\log\frac{a_1}{c}\ .$$

Given that the LHS of the second equation, which determins y_1, is decreasing in y_1, the above system of equation is consistent if and only if:

$$(a_1+N_1-2)(\frac{c}{a_1})^{\frac{a_1-1}{N_1-1}} + N_2 \ge a_1\ .$$

Exit equilibria with $x > 0, y_1 = 0$: This requires that $x = \frac{1}{a_1}\log\frac{a_1}{c}$, and that:

$$\begin{aligned} &-(a_1+N_1-2)\exp(-x) - N_2\exp(-x) + c \ge 0 \\ \Leftrightarrow &(a_1+N-2)(\frac{c}{a_1})^{\frac{1}{a_1}} \le c \Leftrightarrow (1+\frac{N-2}{a_1})^{a_1} \le (\frac{c}{a_1})^{a_1-1}\ . \end{aligned}$$

It is easy to see that the LHS is always greater than 1, while the RHS is less than 1, making this exit equilibrium impossible.

Exit equilibria with $x = 0, y_1 = 0$: This case will never happen, as $c < a_1$.

Exit equilibrium: a user from N_2 leaving

Let x denote the investment of the deviating user from group N_2, and y_1 and y_2 denote the investments of users remaining in N_1 and N_2. We have the following system of equations:

$$\begin{aligned} &-a_2\exp(-a_2x - N_1y_1 - (N_2-1)y_2) + c \\ &\qquad \ge 0\ , \\ &-(a_1+N_1-1)\exp(-x-(a_1+N_1-1)y_1-(N_2-1)y_2) \\ &\quad -(N_2-1)\exp(-x-N_1y_1-(a_2+N_2-2)y_2) + c \\ &\qquad \ge 0\ , \\ &-N_1\exp(-x-(a_1+N_1-1)y_1-(N_2-1)y_2) \\ &\quad -(a_2+N_2-2)\exp(-x-N_1y_1-(a_2+N_2-2)y_2) + c \\ &\qquad \ge 0\ . \end{aligned}$$

By an argument similar to that in the derivation of the SO solution in IV-D, we will always have $y_2 = 0$. As a result, the system of equations reduces to:

$$\begin{aligned} &-a_2\exp(-a_2x - N_1y_1) + c && \ge 0\ , \\ &-(a_1+N_1-1)\exp(-x-(a_1+N_1-1)y_1) && \\ &\qquad -(N_2-1)\exp(-x-N_1y_1) + c && \ge 0\ . \end{aligned}$$

We consider the following possible cases.

Exit equilibria with $x > 0, y_1 > 0$: This would require a solution to the following system of equations:

$$\begin{aligned} &a_2\exp(-a_2x - N_1y_1) && = c\ , \\ &(a_1+N_1-1)\exp(-x-(a_1+N_1-1)y_1) && \\ &\qquad + (N_2-1)\exp(-x-N_1y_1) && = c\ . \end{aligned}$$

From the first equation we know that $a_2x + N_1y_1 = \log\frac{a_2}{c}$. To solve the system, we substitute y in the second equation and obtain:

$$\begin{aligned} &(a_1+N_1-1)\exp(-\log\frac{a_2}{c} - (1-a_2)x - \frac{a_1-1}{N_1}[\log\frac{a_2}{c} - a_2x]) \\ &\qquad + (N_2-1)\exp(-\log\frac{a_2}{c} - (1-a_2)x) = c\ , \end{aligned}$$

which can be solved numerically.

Exit equilibria with $x = 0, y_1 > 0$: This would require:

$$\begin{aligned} &a_2\exp(-N_1y_1) && \le c\ , \\ &(a_1+N_1-1)\exp(-(a_1+N_1-1)y_1) && \\ &\qquad + (N_2-1)\exp(-N_1y_1) && = c\ . \end{aligned}$$

Note that from the equation above we can say:

$$\begin{aligned} \exp(-N_1y_1) &= \frac{c-(a_1+N_1-1)\exp(-(a_1+N_1-1)y_1)}{N_2-1} \\ &\le c \le \frac{c}{a_2}\ . \end{aligned}$$

Therefore, this is always a possible equilibrium.

Exit equilibria with $x > 0, y = 0$: This requires that $x = \frac{1}{a_2}\log\frac{a_2}{c}$, and that:

$$\begin{aligned} &-(a_1+N_1-1)\exp(-x) - (N_2-1)\exp(-x) + c \ge 0 \\ &\Leftrightarrow (a_1+N-2)(\frac{c}{a_2})^{\frac{1}{a_2}} \le c \\ &\Leftrightarrow \frac{a_1+N-2}{c} \le (\frac{a_2}{c})^{\frac{1}{a_2}}\ . \end{aligned}$$

The above holds when a_2 is small (RHS is maximized at $a_2 = c\exp(1)$).

Exit equilibria with $x = 0, y = 0$: This case will never happen, as $c < a_2$.

Exit equilibria of the weighted effort model - single dominant user

In this appendix, we solve for the socially optimal investment profile, and identify the possible exit equilibria, and parameter conditions under which each equilibrium is possible. It is easy to show that in a socially optimal investment profile $\mathbf{x}^*$, only user 1 will be exerting effort, so that:

$$x_1^* = \frac{1}{a}\ln\frac{aN}{c}, \quad x_j^* = 0, \forall j = 2, \dots, N\ .$$

We next find the exit equilibria under two different cases. First, if any non-dominant user $i \ne 1$ steps out of the mechanism, user 1 will continue exerting all effort, but at a lower level given by:

$$\hat{x}_1^i = \frac{1}{a}\log\frac{a(N-1)}{c}, \quad \hat{x}_j^i = 0, \forall j = 2, \dots, N\ .$$

Next, if user 1 steps out of the mechanism, there are two possible exit equilibria: if $a > N-1$, there will be enough externality for users $j \neq 1$ to continue free-riding, resulting in the following equilibrium investment levels:

$$\hat{x}^1_1 = \frac{1}{a}\log\frac{a}{c}, \quad \hat{x}^1_j = 0, \forall j = 2, \dots, N \ .$$

However, when $a < N-1$, user 1 will free-ride on the externality of other users' investments, leading to the exit equilibrium:

$$\hat{x}^1_1 = 0, \quad \hat{x}^1_j = \frac{1}{N-1}\log\frac{N-1}{c}, \forall j = 2, \dots, N \ .$$

BB AND VP IN EXIT EQUILIBRIA - SINGLE DOMINANT USER

In this appendix, we look at the performance of the Pivotal and Externality mechanisms, under the different exit equilibria identified in Section IV-E, summarized in Table II.

In the Externality mechanism, users' taxes are given by:

$$\begin{aligned} t^E_1(\mathbf{x}^*) &= cx^*_1(\frac{1}{N}-1) \\ t^E_j(\mathbf{x}^*) &= \frac{c}{N}x^*_1, \forall j = 2, \dots, N \ . \end{aligned}$$

For non-dominant users $i \in \{2, \dots, N\}$ to voluntarily participate in the mechanism, we require $g_i(\hat{\mathbf{x}}^i) \geq \mathrm{g}_i(\mathbf{x}^*, t^E_i(\mathbf{x}^*))$:

$$\begin{aligned} &\frac{c}{a(N-1)} \geq \frac{c}{aN} + \frac{c}{aN}\log\frac{aN}{c} \\ \Leftrightarrow &\frac{1}{N-1} \geq \log N + \log\frac{a}{c} \ . \end{aligned}$$

However, $\log N \geq \frac{1}{N-1}, \forall N \geq 3$, and $a > c$. Therefore, the voluntary participation constraints will always fail to hold in the Externality mechanism.

Finally, we analyze the total budget in the Pivotal mechanism for the current scenario. The taxes for the non-dominant users $i \neq 1$ will be given by:

$$\begin{aligned} t^P_i &= (N-1)\frac{c}{aN} + cx^*_1 - (N-1)\frac{c}{a(N-1)} - c\hat{x}^i_1 \\ &= \frac{c}{a}(\log\frac{N}{N-1} - 1) \ . \end{aligned}$$

The taxes for user 1 will depend on the realized exit equilibrium. If $a < N-1$, this tax is given by:

$$\begin{aligned} t^P_1 &= (N-1)\frac{c}{aN} - (N-1)\frac{c}{N-1} - c(N-1)\hat{x}^1_j \\ &= (N-1)\frac{c}{aN} - c(1 + \log\frac{N-1}{c}) \ . \end{aligned}$$

The sum of the Pivotal taxes under this parameter conditions will then be given by:

$$\begin{aligned} \sum_i t^P_i = c(&\frac{N-1}{a}(\log\frac{N}{N-1} - 1 + \frac{1}{N}) \\ &- (1 + \log\frac{N-1}{c})) \end{aligned}$$

Note that $\log z - \frac{1}{z} < 0, \forall z < \frac{3}{2}$, and therefore, with $N \geq 3$, the above sum is always negative. We conclude that the Pivotal mechanism will always carry a deficit.

On the other hand, when $a > N-1$, the tax for the dominant user is given by:

$$t^P_1 = (N-1)\frac{c}{aN} - (N-1)\frac{c}{a} = (N-1)\frac{c}{a}(\frac{1}{N} - 1) \ .$$

The sum of the Pivotal taxes will then be given by:

$$\sum_i t^P_i = \frac{c(N-1)}{a}\left(-1 + \log\frac{N}{N-1} - 1 + \frac{1}{N}\right)$$

By the same argument as before, the above sum will always be negative, indicating a budget deficit in the Pivotal mechanism under this scenario as well.