# United We Fall: On the Nash Equilibria of Multiplex Network Games

Raman Ebrahimi
*Electrical and Computer Engineering*
*University of California, San Diego*
raman@ucsd.edu

Parinaz Naghizadeh
*Electrical and Computer Engineering*
*University of California, San Diego*
parinaz@ucsd.edu

***Abstract*—Network games have provided a framework to study strategic decision making processes that are governed by an underlying network of interdependencies among agents. However, existing models do not account for environments in which agents simultaneously interact over multiple networks. In this paper, we propose a model of *multiplex network games* to capture the different modalities of interactions among strategic agents. We then explore how the properties of the constituent networks of a multiplex network can undermine or support the uniqueness of its Nash equilibria. We first show that in general, even if the constituent networks are guaranteed to have unique Nash equilibria in isolation, the resulting multiplex need not have a unique equilibrium. We then identify certain subclasses of networks wherein guarantees on the uniqueness of Nash equilibria on the isolated networks lead to the same guarantees on the multiplex network game. We further highlight that both the largest and smallest eigenvalues of the constituent networks (reflecting their connectivity and two-sidedness, respectively) are instrumental in determining the uniqueness of the multiplex network equilibrium. Together, our findings shed light on the reasons for the fragility of the uniqueness of equilibria in multiplex networks, and potential interventions to alleviate them.**

***Index Terms*—Nash equilibrium, multiplex network, uniqueness, P-matrix, lowest eigenvalue.**

## I. Introduction

Complex networks provide a powerful framework for understanding and analyzing real-world environments where agents do not operate in isolation, but rather interact with and influence one another. In particular, when the agents in these networks are rational and self-interested, the interactions among them can be modeled as a *network game*; see [1] for a survey. Such networked strategic interactions emerge in the studies of local provision of public goods on networks [2]–[4] (such as cyber-security, R&D), spread of shocks in financial markets [5], and pricing in the presence of social effects and externalities [6], [7], to name a few.

A primary goal of the research on network games has been to understand how the structural properties of the network of interactions among the agents influences the equilibrium outcomes. This understanding can help us interpret how a network's performance metrics are affected by its structure, plan targeted policy interventions or economic incentives to shape agents' behavior, and design/modify network structures to attain desirable equilibrium outcomes. In particular, existing works have identified necessary and sufficient conditions for the existence, uniqueness, and/or stability of Nash equilibria of network games [2]–[4], [8]–[20].

Although these existing works capture a network of interactions between the agents, they fail to capture *the various networks* over which the agents interact. For instance, individuals are influenced by information received over multiple social networks, as well as face-to-face interactions, when making decisions. Similarly, firms in a market can cooperate and compete with each other through different modes of business (e.g., both physical storefronts and online shops). In these situations, a *multiplex network* model can be used to simultaneously account for the multiple networks of interactions among the agents. The study of multiplex networks, as opposed to the study of its constituent *single-layer* networks in isolation, can offer a nuanced understanding of how different modalities of information and interactions impact agents' behavior and network-level outcomes. While existing work has provided such insights (primarily about percolation and spread of dynamical processes on multiplex networks), the study of games on this type of multilayer networks remains largely unexplored, with game-theoretical modeling and analysis identified as an open area of research by surveys of the field [21]–[24].

Motivated by this, in this paper, we extend the existing models of single-layer network games, and propose a *multiplex network game* to study strategic interactions over multiplex networks. We then use this model to identify conditions under which the Nash equilibria (NE) of multiplex network games are (not) unique based on the properties of its constituent single-layer networks. These findings uncover potential reasons for the fragility of the uniqueness of Nash equilibria in multiplex networks.

### A. Paper overview and contributions

We study a two-layer multiplex network consisting of two single-layer networks $\alpha$ and $\beta$. In a network game, the utility of an agent is assumed to be a function of a weighted sum of its own effort and its neighboring agents' efforts, with the intensity and nature of the agents' influences on each other captured in a directed and weighted *interdependency matrix*; denote these by $A$ and $B$ for layers $\alpha$ and $\beta$, respectively.

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In our proposed model of the multiplex network game (Section II), agents are influenced by both of these interdependency matrices simultaneously, experiencing a weighted interdependency matrix $G = \kappa A + (1-\kappa)B$ in the multiplex network game, where $\kappa \in [0,1]$ is the relative importance of layer $\alpha$.

Prior work on single-layer network games in [4] shows that the network game played on a network with (a weighted and directed) interdependency matrix $M$ will have a unique Nash equilibrium if and only if $I+M$ is a P-matrix.[1] Our first contribution is to identify conditions under which $I+G$, the weighted sum of $I+A$ and $I+B$, is (and is not) a P-matrix (Section III). We first provide an example (Example 1) to show that the sum of two P-matrices is in general *not* a P-matrix; this means that even if the two layers are guaranteed to have unique Nash equilibria in isolation, the resulting multiplex need not have a unique NE. We then identify a sufficient condition (Proposition 1) under which the multiplex game is not guaranteed to have a unique NE (that is, there will be payoff realizations under which the multiplex either has no NE or has multiple NE). We also show (Proposition 2) that for certain subclasses of networks, if the games on the isolated layers are guaranteed to have a unique NE, so will the multiplex network. Intuitively, the identified networks (namely, strictly row diagonally dominant networks and B-networks) are such that the influence of agents on each other can be sufficiently bounded.

We then consider the special case of symmetric (i.e., undirected) networks (Section IV). For this case, prior works on single-layer networks [2], [4] show that the game on a network with symmetric interdependency matrix $M$ will have a unique Nash equilibrium if and only if $|\lambda_{\min}(M)| < 1$. The lowest eigenvalue is a measure of a network's "two-sidedness", with a smaller (more negative) lowest eigenvalue being an indication that agents' actions "rebound" more in the network [2]. This means that, intuitively, this existing condition on NE uniqueness requires the (single-layer) network to have limited rebounds or two-sidedness.

Extending this to multiplex network games, we first note that in contrast to the general case of Section III, if two symmetric layers $\alpha$ and $\beta$ already have structures that are conducive to unique NE in isolation, so will the symmetric multiplex network emerging from joining them. In light of this, we identify situations in which layer $\alpha$ supports a unique NE, yet layer $\beta$ can undermine the guarantee on the uniqueness of NE of the resulting multiplex. In particular, we show (Proposition 3) that if the connectivity of layer $\alpha$ (as characterized by its largest eigenvalue) is not high enough to mute the ups and downs introduced by layer $\beta$ (as characterized by its smallest eigenvalue), then the multiplex is not guaranteed to have a unique Nash equilibrium. We further elaborate on this observation in some special classes of networks (namely, regular, tree, random, and scale-free graphs).

[1] A P-matrix, which most notably emerges in the study of linear complementarity problems (LCPs) [25], encompasses positive definiteness as a special case; we elaborate on this in Section II-C.

### B. Related work

Our work is at the intersection of two lines of literature: (i) the study of properties of Nash equilibria of single-layer network games, and (ii) game-theoretical modeling and analysis on multilayer and multiplex networks.

*Nash equilibria of single-layer network games.* Our work is most closely related to those studying the existence, uniqueness, and/or stability of Nash equilibria of games with linear [2], [4], [10]–[12] and nonlinear [15]–[17] best-replies played on single-layer networks. Our proposed model of multiplex network games is an extension of those of single-layer network games with linear best-replies, and as such extends the condition on the uniqueness of NE in these games. In particular, we show that not only the lowest eigenvalue of the individual layers (which has been shown to be of importance in guaranteeing uniqueness of NE in single-layer network games of both linear and non-linear best-replies), but also their *largest eigenvalues*, play a role in determining the uniqueness of the NE of a multiplex game.

*Games on multilayer/multiplex networks.* While game theoretical decision making on multilayer networks has also been studied in some prior works [21], [26]–[28], the majority of the research has focused on evolutionary games and the emergence of cooperation in public good games. In particular, [27] studies the problem of emergence of cooperation in multiplex networks, and shows that a multiplex structure enhances the resilience of cooperation through a nontrivial organization of the cooperative behavior across layers. The work of [28] studies similar cooperation games, and explores the impact of the number of layers and the number of links on the multiplex network's ability to support cooperation over defection. These works have both considered a binary action (cooperate/defect) game, played on specific classes of graphs. In contrast, we propose a more general class of multiplex networks games, and provide insights into their Nash equilibria properties.

The recent work of [29] is also related to ours, as it proposes a *multi-relational* network game which can be interpreted as a model of games played on a multilayer network. In the multi-relational network game, each agent has a multi-dimensional action space, with the utility from each action dimension being governed by a different network of interdependencies. The main focus of [29] is on identifying *summary representations* of the game matrices that can be used to significantly lower the *computation complexity* of ascertaining the uniqueness of NE. In addition to our different model of multiplex network games, our work is different in its focus: we illustrate how the properties of the constituent layers of a multiplex network can undermine or support the uniqueness of the NE of the multiplex network game.

## II. Model and Preliminaries

### A. Single-layer network games

We consider a set of $N$ agents, initially interacting with each other over an incumbent network $\alpha$ (e.g., an existing cyber-physical system, social network, or industry). This network is specified by a graph $\mathcal{G}_\alpha :=< \mathcal{V}, A >$, where the $N$ agents constitute the set of vertices $\mathcal{V}$, and $A$ is the weighted and directed *interdependency matrix* over network $\alpha$.

Each agent $i$ selects an *effort* level $x \in \mathbb{R}_{\geq 0}$; this could represent the amount of investment in a public good such as cyber security or R&D. The agent's utility is determined by its own effort, as well as the effort of its neighboring agents in the network.

Specifically, an edge $a_{ij} \in A$ indicates that agent $i$ is affected by agent $j$'s effort. If $a_{ij} > 0$ (respectively, $< 0$), we say agent $j$'s effort is a *substitute* (respectively, *complement*) to agent $i$'s effort. In our setting, a strategic substitute (resp. complement) means that effort by agent $j$ provides positive (resp. negative) externality to agent $i$, in that an increase in effort by agent $j$ allows agent $i$ to decrease its own effort (resp. requires agent $i$ to increase its effort) and still receive the same overall payoff. For instance, security investments can be a strategic substitute when a better protected firm $j$ positively impacts other firms $i$ that share operations and assets with firm $j$, by decreasing the risk of business interruption or asset compromise. On the other hand, security investments can be a strategic complement when an increase in firm $j$'s protection makes a similar, but less protected firm $i$ a more attractive target for attackers.

Formally, let $\mathbf{x} = \{x_1, x_2, \dots, x_N\}$ denote the vector of all agents' efforts. Agent $i$'s utility in network $\alpha$ is given by:

$$u_i(\mathbf{x}; A) = b_i(x_i + \sum_j a_{ij} x_j) - c_i x_i \ , \tag{1}$$

where $b_i(\cdot) : \mathbb{R} \to \mathbb{R}$ is a twice-differentiable, strictly increasing, and strictly concave *benefit function*, and $c_i > 0$ is the unit cost of effort for agent $i$.

The (single-layer) network game specified by the set of $N$ agents, their efforts $\mathbf{x}$, and their utility functions $\{u_i(\mathbf{x}; A)\}$ has been studied in prior works (e.g., [2], [4], [9]). In particular, these games are known as games of *linear best-replies*, as the Nash equilibrium $\mathbf{x}^*$ is determined by a set of linear best-response equations of the form:

$$x_i^* = \max\{0, q_i - \sum_j a_{ij} x_j^*\} \ , \tag{2}$$

where $q_i$ satisfies $b_i'(q_i) = c_i$. Intuitively, an agent $i$ wants to receive an aggregate effort level $q_i$ at equilibrium; this is the effort level at which the agent's marginal benefit and marginal cost of effort are equalized. The best-response (2) states that the agent exerts the effort $x_i^*$ that will allow it to reach an aggregate effort level $q_i$ given the spillover $\sum_j a_{ij} x_j^*$ received from its neighboring agents' effort at equilibrium (or exerts no effort if the spillovers already provide aggregate effort $q_i$ or higher).

### *B. Multiplex network games*

Our goal in this paper is to contrast the properties of the NE of the single-layer network $\alpha$, with that of a two-layer multiplex network emerging after the addition of a new network $\beta$ of interactions among the agents (e.g., a new CPS, social network, or industry). Let this second layer be defined by a graph $\mathcal{G}_\beta :=< \mathcal{V}, B >$, with the same set of vertices as network $\alpha$, but its own interdependency matrix $B$.

The two-layer multiplex network $\mathcal{G} :=< \mathcal{N}, \{A, B\} >$ is the environment in which interactions between the $N$ agents occur over both networks $\alpha$ and $\beta$ simultaneously, but each governed by a different interdependency matrix. The utility of agent $i$ in the multiplex network is given by

$$\begin{aligned} &u_i(\mathbf{x}; A, B, \kappa) = \\ &\quad b_i(x_i + \kappa \sum_j a_{ij} x_j + (1-\kappa) \sum_j b_{ij} x_j) - c_i x_i, \end{aligned} \tag{3}$$

where $\kappa \in [0, 1]$ captures the effect of each layer on the agent's utility, with higher $\kappa$'s indicating higher effects from the incumbent network $\alpha$.

The resulting *multiplex network game* is again a game of linear best-replies, where at equilibrium, agent $i$ aims to choose $x_i^*$ so as to reach the same aggregate level of effort $q_i$, but this time while being exposed to spillovers $\kappa \sum_j a_{ij} x_j^* + (1-\kappa) \sum_j b_{ij} x_j^*$ from the multiplex network. As such, the multiplex network game can be viewed as a network game played over the interdependency matrix $G := \kappa A + (1-\kappa) B$.

### *C. Uniqueness of NE on single-layer network games*

By exploring the connection of the Nash equilibrium problem and linear complementarity problems (LCPs), [4] identified conditions for existence and uniqueness of the NE of single-layer network games. In particular, we begin with the following definition.

**Definition 1.** *A square matrix $M$ is a* P-matrix *if the determinants of all its principal minors (i.e., the square submatrix obtained from $M$ by removing a set of rows and their corresponding columns) are strictly positive.*

The class of P-matrices includes positive definite (PD) matrices as a special case;[2] in particular, every PD matrix (whether symmetric or not) is a P-matrix, but there are (asymmetric) P-matrices that are not PD [25]. We also note that for symmetric matrices, the two notions are equivalent, i.e., a symmetric matrix is a P-matrix if and only if it is PD.

The following theorem provides the necessary and sufficient condition for the Nash equilibrium of the network game on a single-layer network (whether symmetric or not) to be unique.

**Theorem 1.** *[4, Theorem 1] The single-layer network game on a network with interdependency matrix $A$ has a unique Nash equilibrium if and only if $I + A$ is a P-matrix.*

The following corollary is the special case of Theorem 1 for symmetric networks.

**Corollary 1.** *[2], [4, Corollary 1] Consider a single-layer network game on a symmetric interdependency matrix $A$.*

[2]A common convention adopted in some of the literature is to define positive definiteness for symmetric (or Hermitian) matrices, owing to their roots in quadratic forms. However, we adopt the more general definition here: A square matrix $M$ (whether symmetric or not) is positive definite if $x^T M x > 0$ for all $x \neq 0$.

*This game has a unique Nash equilibrium if and only if $|\lambda_{\min}(A)| < 1$.*

Bramoullé, Kranton, and D'amours [2] were the first to show that the above condition on the lowest eigenvalue of (symmetric) interdependency matrices is sufficient for the uniqueness of NE of network games; Naghizadeh and Liu [4] further showed that this condition is necessary.

We can use Theorem 1 and its corollaries to discuss the existence and uniqueness of the NE of a multiplex network game in terms of the properties of its interdependency matrix $G = \kappa A + (1-\kappa)B$. In particular, the game has a unique Nash equilibrium if and only if $I+G$ is a P-matrix, and if and only if $|\lambda_{min}(G)| < 1$ when $G$ is symmetric.

While we can directly check these conditions for $G$, it might be of interest for computational efficiency, and also for gaining intuition about the operation of multiplex networks (as we show shortly), to identify when these conditions on $G$ are true, or when they fail to hold, based on the properties of $A$ and $B$. Specifically, we want to understand when $I+G$, the weighted sum of two matrices $I+A$ and $I+B$, is (not) a P-matrix, as well as bounds on $G$'s minimum eigenvalue in terms of the spectrum of $A$ and $B$. We present such conditions in the remainder of the paper.

## III. Nash Equilibria of Multiplex Networks with General Interdependency Matrices

We first consider games with general (directed) interdependency matrices, and ask when $I+G$, the weighted sum of $I+A$ and $I+B$, is a P-matrix? This will require us to check that the determinants of all principal minors of $I+G$ are positive; these are the determinants of the sum of the corresponding principal minors in $I+A$ and $I+B$. However, the determinant of the sum of two square matrices $A$ and $B$ is in general not expressible in terms of the determinants of the two matrices.[3] This means that knowledge of the P-matrix property of $I+A$ and/or $I+B$ does not necessarily help establish the P-matrix property for $I+G$.

In fact, the following example shows that the sum of two P-matrices is *not* always a P-matrix.

**Example 1.** *Consider a two-agent multiplex network game, with the (asymmetric) interdependency matrices $A = \begin{pmatrix} 0 & a-\epsilon \\ \frac{1}{a} & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b-\epsilon \\ \frac{1}{b} & 0 \end{pmatrix}$, Provided that a, b, and $\epsilon$ are positive, $I+A$ and $I+B$ are P-matrices.*

*Once the two layers are connected, the matrix $G$ for the multiplex network game is given by $G = \begin{pmatrix} 0 & \kappa a + (1-\kappa)b - \epsilon \\ \frac{\kappa}{a} + \frac{1-\kappa}{b} & 0 \end{pmatrix}$. Then, $I+G$ is a P-matrix if and only if*

$$\kappa^2 + (1-\kappa)^2 + \kappa(1-\kappa)(\frac{a}{b} + \frac{b}{a}) - \epsilon(\frac{\kappa}{a} + \frac{1-\kappa}{b}) < 1 \ .$$

*However, the left-hand side of the inequality above will be increasing in $a$ once $a$ is sufficiently large. Therefore, there exists an $\bar{a}$ such that for $a \geq \bar{a}$, $I+G$ will not be a P-matrix.*

*To further illustrate, assume the game has benefit function $b_i(x) = 1 - \exp(-x)$, and unit costs of effort $c_1 = \frac{1}{e}$ and $c_2 = \frac{1}{\sqrt{e}}$. Let $\epsilon = 1$, $b = 1$, $\kappa = 0.5$. Then, layer $\alpha$ has a unique Nash equilibrium; when $a > 3$, this unique NE is $\mathbf{x}^* = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$.*

*For the multiplex, $I+G = \begin{pmatrix} 1 & \frac{1}{2}(a-1) \\ \frac{1}{2}(\frac{1}{a}+1) & 1 \end{pmatrix}$. This is a P-matrix if and only if $a \leq 2+\sqrt{5}$. In particular,*

- *If $a=4$, $I+G$ is a P-matrix, and the multiplex game will have a unique Nash equilibrium $\mathbf{x}^* = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$.*
- *If $a=5$, $I+G$ is not a P-matrix, and the multiplex has two Nash equilibria: $\mathbf{x}^* = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$ or $\mathbf{x}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.*[4]

To see why this example emerges, note that $a$ captures the influence of agent 2 on agent 1 in network $\alpha$. This single-layer network is structured such that when agent 2's influence on agent 1 increases, the reverse influence from agent 1 on agent 2 decreases proportionally, so that the game matrix can remain a P-matrix. In contrast, once the two networks are joined, when $a$ increases, part of the reverse influence, $\frac{1-\kappa}{b}$, is constant and does not respond to the increase in $a$. This can break the P-matrix property of the game matrix, and as such, the guarantee on the equilibrium uniqueness. In this example, it leads to the dependence of $j$ on $i$ to increase along with the dependence of $i$ on $j$, reaching a level that allows either agent to free-ride on the other's effort in the multiplex game, and resulting in the two possible NE.

### A. When is $I+G$ not a P-matrix?

We now generalize the intuition from Example 1 to identify conditions under which $I+G$ is not a P-matrix. In particular, note that $I+G$ is a P-matrix if and only if the determinant of *all* its principal minors are positive. The following proposition identifies conditions under which at least one of the principal minors has a non-positive determinant.

**Proposition 1.** *Let $M^l_{ij}$ be the $2\times 2$ minor obtained by removing all rows and columns except $i$ and $j$ from the interdependency matrix of layer $l$. If there exists a pair of agents $i$ and $j$ such that*

$$\frac{a_{ij}}{b_{ij}}(1-\det(M^\alpha_{ij})) + \frac{b_{ij}}{a_{ij}}(1-\det(M^\beta_{ij})) \geq 2 + \frac{\kappa}{1-\kappa}\det(M^\alpha_{ij}) + \frac{1-\kappa}{\kappa}\det(M^\beta_{ij}) \ , \quad (4)$$

[3] The *Marcus–de Oliveira determinantal conjecture*, which conjectures that the determinant of the sum of two matrices is in a convex hull determined by the eigenvalues of the two matrices, remains as one of the open problems in matrix theory, with the conjecture shown to hold for some special classes including Hermitian matrices [30].

[4] Note that the P-matrix condition is a guarantee that the network structure will lead to an NE that is unique *independent* of the realizations of benefit and cost functions. When the condition fails to hold, there can still be benefit and cost realizations under which the NE is unique. For instance, in this example, if we had $c_2 = \frac{1}{e}$, the multiplex with $a=5$ would have also had a unique NE $\mathbf{x}^* = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$.

*then the multiplex network is not guaranteed to have a unique Nash equilibrium. That is, there are cost and benefit functions for which the multiplex network either does not have a Nash equilibrium, or has multiple Nash equilibria.*

*Proof:* Consider any pair of agents, $i$ and $j$. The principal minor corresponding to this pair in the multiplex network is $M_{ij} := \begin{pmatrix} 1 & \kappa a_{ij} + (1-\kappa) b_{ij} \\ \kappa a_{ji} + (1-\kappa) b_{ji} & 1 \end{pmatrix}$. A necessary condition for $I+G$ to be a P-matrix is for this principal minor to have a positive determinant. We can write this condition in terms of the determinants of the pair of agents' corresponding principal minors $M_{ij}^{\alpha}$ and $M_{ij}^{\beta}$ in each layer, as follows:

$$
\begin{aligned}
&\det(M_{ij}) > 0 \\
\Leftrightarrow\ & \kappa^2 a_{ij} a_{ji} + (1-\kappa)^2 b_{ij} b_{ji} + \kappa(1-\kappa)(a_{ij} b_{ji} + b_{ij} a_{ji}) < 1 \\
\Leftrightarrow\ & a_{ij} b_{ji} + b_{ij} a_{ji} < 2 + \frac{\kappa}{1-\kappa} \det(M_{ij}^{\alpha}) + \frac{1-\kappa}{\kappa} \det(M_{ij}^{\beta}) \\
\Leftrightarrow\ & \frac{a_{ij}}{b_{ij}} (1 - \det(M_{ij}^{\alpha})) + \frac{b_{ij}}{a_{ij}} (1 - \det(M_{ij}^{\beta})) \\
& \qquad < 2 + \frac{\kappa}{1-\kappa} \det(M_{ij}^{\alpha}) + \frac{1-\kappa}{\kappa} \det(M_{ij}^{\beta}) \qquad (5)
\end{aligned}
$$

In particular, if (5) is violated for even one pair of agents $i$ and $j$, the multiplex will not be guaranteed to have a unique Nash equilibrium. ■

It is worthwhile to again note that even if both layers $\alpha$ and $\beta$ satisfy the P-matrix condition, so that the sub-determinants $\det(M_{ij}^{l})$ are positive, the condition (4) can still hold at sufficiently large $a_{ij}$ (note that the determinant term can be kept constant by adjusting $a_{ji}$ accordingly).

Proposition 1 could also be extended to state conditions on higher order principal minors (e.g., highlighting that non-uniqueness can be caused by the change in the nature and intensity of interactions among a set of three agents).

### *B. When is $I+G$ a P-matrix?*

While as shown in Example 1 and Proposition 1, the sum of two P-matrices is in general not a P-matrix, there are specific subclasses of P-matrices which are closed under summation, as shown in the following proposition.

**Proposition 2.** *For any of the following cases, $I+G$ where $G = \kappa A + (1-\kappa) B$, $\kappa \in [0,1]$, will be a P-matrix.*

1) *$A$ and $B$ are* symmetric *matrices.*
2) *$I+A$ and $I+B$ are* strictly row diagonally dominant*, i.e., $\sum_{j \neq i} |a_{ij}| < 1$ and $\sum_{j \neq i} |b_{ij}| < 1$, $\forall i$.*
3) *$I+A$ and $I+B$ are* B-matrices*, i.e., $1+\sum_j a_{ij} > 0$ and $\frac{1}{N}(1 + \sum_{j \neq k} a_{ij}) > a_{ik}$, $\forall i$ and $\forall k \neq i$, and similarly for $B$.*

*Proof:*

1) This is true because a symmetric matrix is a P-matrix if and only if it is positive definite [25], and the sum of positive definite matrices is positive definite.
2) By the triangle inequality, $|\kappa a_{ij} + (1-\kappa) b_{ij}| < \kappa |a_{ij}| + (1-\kappa)|b_{ij}|$. Therefore, we have $\sum_{j \neq i} |\kappa a_{ij} + (1-\kappa) b_{ij}| < 1$, $\forall i$, and as such, $I+G$ is strictly row diagonally dominant as well. by the Gershgorin circle theorem, all real eigenvalues of a strictly row diagonally dominant matrix with positive diagonal elements are positive. The determinant of a matrix is the product of its eigenvalues, and as for real matrices, the complex eigenvalues appear in pairs with their conjugates, $I+G$ has a positive determinant. The same argument holds for all principal minors of $I+G$. Therefore, $I+G$ is a P-matrix.
3) A B-matrix is a subclass of P-matrices [31]. It is easy to check that given that $I+A$ and $I+B$ are B-matrices, $I+G$ also satisfies the conditions of a B-matrix, and is therefore a P-matrix.

■

*Intuition.* We delve deeper into the case of symmetric matrices in the next section. The remaining two cases, row diagonally dominant and B-matrices, set limits on the influence of agents on each other. Proposition 2 notes that these limits will carry over when two networks connect with each other. In particular, a row diagonally dominant matrix limits the cumulative maximum influence of neighboring agents on an agent $i$'s utility, relative to the agent's self-influence (here, normalized to 1). If the externalities received from other agents are limited in both layers, they will also be limited when layers are interconnected. B-matrices on the other hand require that the row averages dominate any off-diagonal entries, meaning that no one neighbor's externality on agent $i$'s utility is higher than the average of all the other influences the agent experiences (both self-influence and the externality from the remaining neighbors). Again, if this is true in both layers, it will remain true when the two layers are interconnected as well.

## IV. Nash Equilibria of Multiplex Networks with Symmetric Interdependency Matrices

We now turn to the special case of symmetric (undirected) networks. We begin by noting that the sum of two positive definite matrices is a positive definite matrix. That is, in contrast to the general case of Section III, if we know that two symmetric layers $\alpha$ and $\beta$ already have structures that are conducive to unique NE, so will the symmetric multiplex network emerging from joining them. In light of this, we focus on identifying situations in which layer $\alpha$ supports a unique NE, yet the introduction of layer $\beta$ undermines the guarantee on the uniqueness of NE of the resulting multiplex.

Specifically, we ask: when is $|\lambda_{\min}(G)| \geq 1$? We use the well known Weyl's inequalities to answer this question, which provides bounds on the eigenvalues of the sum of two matrices in terms of the eigenvalues of the constituent matrices.

*Weyl's inequalities [32]:* Let $H = H_1 + H_2$, $H_1$, and $H_2$ be $n \times n$ Hermitian matrices, with their respective eigenvalues $\lambda_i$ indexed in decreasing order, i.e., $\lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \ldots \geq$

$\lambda_n = \lambda_{\min}$. Then, the following inequalities hold:

$$\lambda_j(H_1) + \lambda_k(H_2) \leq \lambda_i(H) \leq \lambda_r(H_1) + \lambda_s(H_2)$$
$$\text{s.t.} \quad j + k - n \geq i \geq r + s - 1 \ .$$

We now use these to identify conditions under which the multiplex is not guaranteed to have a unique NE.

**Proposition 3.** *If*

$$|\lambda_{\min}(B)| \geq \frac{1}{1-\kappa}\left(1 + \kappa\lambda_{\max}(A)\right) \ ,$$

*then the multiplex network game is not guaranteed to have a unique Nash equilibrium.*

*Proof:* Since by Corollary 1 we only require a bound on the minimum eigenvalue of $G$, we consider Weyl's inequalities at $i = n$, for $G = \kappa A + (1 - \kappa B)$. These are

$$\lambda_{\min}(G) \leq \min_{r,s, \text{ s.t. } r+s-1 \leq n} (\kappa\lambda_r(A) + (1-\kappa)\lambda_s(B)) \ . \quad (6)$$

We now note that $tr(G) = 0$, and therefore $\lambda_{\min}(G) < 0$. As a result, $|\lambda_{\min}(G)| \leq 1$ is the same as identifying conditions under which $\lambda_{\min}(G) \leq -1$. Consider the term in the minimum upperbound of (6) attained at $\{r = n, s = 1\}$:

$$\lambda_{\min}(G) \leq \kappa\lambda_{\max}(A) + (1-\kappa)\lambda_{\min}(B) \ .$$

If the upperbound above is less than $-1$, then $\lambda_{\min}(G) < -1$, and the multiplex will not be guaranteed to have a unique NE. Re-arranging the inequality, and noting that $\lambda_{\min}(B) < 0$ and $\lambda_{\max}(A) > 0$ (as the traces for both of these matrices, and therefore the sum of their eigenvalues, is equal to zero), leads to the statement of the proposition. ■

*Intuition:* The work of Bramoullé, Kranton, and D'amours [2] was the first to identify that the lowest eigenvalue of a symmetric network has connections to the uniqueness of Nash equilibria of games played on that network. In particular, [2] notes that the lowest eigenvalue is a measure of the network's "two-sidedness", with a smaller (more negative) lowest eigenvalue being an indication that agents' actions rebound more in a network. It is therefore expected that a layer $\beta$ with a large $|\lambda_{\min}(B)|$ will introduce similar effects in the multiplex network. The condition in Proposition 3 shows that this is indeed the case: when network $\beta$ is significantly two-sided, it can undermine the uniqueness of the equilibrium of the multiplex network. Also, as expected, for large $\kappa$ (when layer $\beta$ is less important in determining agents' payoffs), $\lambda_{\min}(B)$ will have less influence on the NE uniqueness.

More interestingly, the severity of rebound effects due to network $\beta$ (its $\lambda_{\min}$) are compared against the extent of *connectivity* of network $\alpha$ (its largest eigenvalue $\lambda_{\max}$). In words, Proposition 3 states that if the connectivity of layer $\alpha$ (as characterized by its largest eigenvalue) is not high enough to mute the ups and downs introduced by layer $\beta$ (as characterized by its smallest eigenvalue), then the multiplex will have either no equilibrium or multiple equilibria for some game instances.

### *A. Special cases: regular, tree, random, and scale-free graphs*

It is known that for a $d$-regular network $M$ (a network where every node has degree $d$), we have $\lambda_{\max}(M) = d$, and $\lambda_{\min}(M) \geq -d$, with equality when the network is bipartite and triangle-less [33]. Assume that the new layer $\beta$ is one such $d_\beta$-regular, triangle-less network, so that $\lambda_{\min}(\beta) = -d_\beta$. For simplicity, set $\kappa = \frac{1}{2}$. We now vary network $\alpha$ to further elaborate on Proposition 3.

*a) Regular network:* If layer $\alpha$ is a $d_\alpha$-regular network, the condition in Proposition 3 reduces to:

$$d_\beta \geq 2 + d_\alpha \ .$$

This means that if the new layer $\beta$ surpasses the incumbent layer $\alpha$ by two degrees (so that it has sufficiently more edges than layer $\alpha$), the new layer can undermine the uniqueness of the multiplex.

*b) Tree network:* For a tree network, $\lambda_{\max}(A) \geq \sqrt{\Delta}$ where $\Delta$ is the largest vertex degree, with the minimum value attained for the star $K_{1,\Delta}$ [34]. Therefore, if layer $\alpha$ is a $K_{1,d_\alpha}$ star network, the condition in Proposition 3 reduces to:

$$d_\beta \geq 2 + \sqrt{d_\alpha} \ .$$

We observe that layer $\beta$ can more easily force the multiplex out of having a unique NE when $\alpha$ is a star network than when it is a regular network. Intuitively, this is because a star graph has lower connectivity than a regular graph, and therefore has a lower capacity to mute the rebounds introduced by layer $\beta$.

*c) Random network:* For a non-sparse random network $G(N,p)$, with a constant $p$, $\lambda_{\max}(G(N,p))$ has a normal distribution with expected value $(N-1)p-(1-p)$ and variance $2p(1-p)$ [35]. Assume that layer $\alpha$ is a random network with the above average maximum eigenvalue. Then, the condition in Proposition 3 reduces to:

$$d_\beta \geq 2 + (N-1)p - (1-p) \ .$$

This indicates that higher link probability $p$ and larger potential number of neighbors $N$ make layer $\alpha$ more connected, and make it harder to ascertain that layer $\beta$ can undermine the multiplex NE uniqueness.

*d) Scale-free network:* Assume next that layer $\alpha$ is a scale-free network; these are networks in which the degree distribution follows a power law, and have been argued to provide close descriptions of real-world networks [36], [37]. From [38], we know that for a scale-free network $\alpha$, $\lambda_{\max}(A) \sim N^{1/4}$ at large $N$. Then, the condition in Proposition 3 reduces to:

$$d_\beta \geq 2 + N^{1/4}.$$

Given this, if $d_\beta = O(N^k)$ for $k > 0.25$, layer $\beta$ can overtake the connectivity of the scale-free network $\alpha$. Further, comparing the conditions for scale-free and random networks $\alpha$, we can see that the right-hand side of the condition in the random network case grows much faster. Intuitively, this is because in random networks nodes have comparable degrees (and when all of them grow with $N$, this increases the network

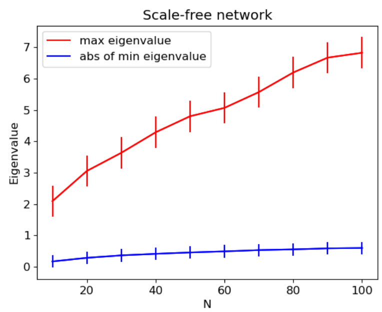


Fig. 1. Change in the maximum and absolute of minimum eigenvalues of a scale-free network as the number of nodes grows.

connectivity); in contrast, the degree increase in scale-free networks when $N$ increases can be due to the emergence of hubs, which do not offer the same extent of connectivity.

Finally, assume both layers $\alpha$ and $\beta$ are scale-free. Figure 1 illustrates the changes in the (size of the) minimum and maximum eigenvalues of a scale-free network as a function of its number of nodes $N$, averaged over 100,000 randomly generated scale-free networks. We conclude that when large scale-free networks are joined into a multiplex network, it is harder to ascertain that one layer can undermine the multiplex NE uniqueness, as the lowest eigenvalue grows much slower than the largest eigenvalue.

## V. Conclusion and Discussion

We have proposed a *multiplex network game* to study networked strategic interactions when agents are affected by different modalities of information and interactions simultaneously. This model enabled us to explore how the properties of the constituent networks undermine or support the uniqueness of the Nash equilibrium of multiplex games. At a technical level, answering these questions required us to understand how the determinant (and lowest eigenvalue) of the sum of two matrices relates to the determinants (and eigenvalues) of the two matrices; neither the determinant nor the eigenvalues of the sum of two matrices have closed-form expressions in general. Our results have therefore leveraged existing inequalities/bounds (e.g., Weyl's inequality) to find (sufficient) negative results, and provided positive answers for special matrix subclasses.

In particular, we have shown that even if the constituent networks are guaranteed to have unique Nash equilibria in isolation, the resulting multiplex need not have a unique Nash equilibrium. We have also identified certain subclasses of networks wherein guarantees on the uniqueness of Nash equilibria on the isolated networks lead to the same guarantees on the multiplex network game. These include row diagonally dominant and B-matrices, both of which set limits on the influence of agents on each other. We further showed that not only the lowest eigenvalue of the individual layers (which has been shown to be of importance in guaranteeing uniqueness of NE in single-layer network games), but also their largest eigenvalues, play a role in determining the uniqueness of the NE of a multiplex game.

Together, our findings shed light on the reasons for the fragility of the uniqueness of equilibria in multiplex networks. They can also provide potential interventions to alleviate them. For instance, we noted that the connectivity of one layer (as characterized by its largest eigenvalue) needs to be high enough to mute the ups and downs introduced by another layer as a necessary condition for equilibrium uniqueness. This suggests that a policy designer could focus their interventions on increasing the connectivity of one layer of a multiplex (e.g., one social network or industry) in an effort to mute the "bipartite-ness" introduced by another.

Moving beyond uniqueness, existing work [2], [15], [17] have argued that the lowest eigenvalue of a single-layer network can also help characterize the *stability* of its Nash equilibrium against different forms of perturbations. A similar exploration of the stability of NE of multiplex networks, as well as the study of multiplex network games with non-linear best-responses, remain as directions of future work.

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