Provision of non-excludable public goods on networks: from equilibrium to centrality measures

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***Abstract*— We consider the provision of non-excludable public goods on a network of interdependent strategic users. We study three different equilibria of these games, namely the Nash equilibrium, socially optimal, and exit equilibrium profiles. We identify properties of the interdependence graph that guarantee the existence and uniqueness of these equilibria. We further establish a connection between users' centralities in their interdependence network, and their efforts at different interior equilibria of these games. These characterizations separate the effects of incoming and outgoing dependencies, as well as the influence of paths of different length, on users' effort levels. We discuss some conceptual and practical implications of this centrality-effort connection.**

I. Introduction

Consider a social or economic network of interdependent entities, whose actions determine the level of provision of a non-excludable public good. Examples include neighborhood levels in cities deciding on creation of new parks or libraries [1], neighboring towns determining whether to invest in measures for reducing pollution [2], businesses in an industry choosing their R&D expenditure [3], and interdependent organizations investing in security measures (whether physical security [4] or cyber-security [5]).

These instances are similar in that they concern the provision of a good (a public project, clean air, research and innovation, and security, respectively) by entities whose investments in the good not only benefit themselves, but also provide positive externalities to (some or all) other users in their network. It is well-known [6] that when users are strategic, the level of provision of these public goods is far from its socially optimal state, most often due to under-investment by users. This is because strategic users do not account for the effect of their choices on the welfare of their network, and may further reduce investments by relying on the externality of others' actions. Optimal provision of public goods therefore calls for the design of appropriate regulations, policies, and/or incentive mechanisms.

The design of policies or incentives in the aforementioned examples is further complicated by the fact that the positive externalities resulting from the provision of the goods are *non-excludable*.[1] That is, even if a user does not cooperate in improving the provision of the public good, she can still benefit from the externality generated by her peers in the network. With this in mind, to understand the provision of non-excludable public goods, and to design adequate incentive mechanisms for achieving their optimal levels, we need to study users' actions at the state of anarchy, as well as their social welfare maximizing actions, and their choices when deviating from any proposed mechanism.

To this end, in this paper, we consider *weighted effort* public good provision games, in which users' benefits from the public good are determined by a weighted sum of their own effort and the efforts of their neighbors in the network. This model is commonly adopted (see Section V), and is of particular interest as it can capture varying degrees and possible asymmetries in the influence of users' efforts on one another. We study the existence and uniqueness of three different effort profiles in these games, namely the Nash equilibrium, socially optimal efforts, and exit equilibria (defined shortly).

Specifically, we provide a characterization of users' social welfare maximizing efforts in these games; these profiles can be implemented using appropriate incentive mechanisms, e.g., the Pivotal (VCG) or Externality mechanisms [10]. In conjunction with such mechanisms, it is of interest to analyze users' voluntary participation constraints. We emphasize that given the non-excludable nature of the studied goods, voluntary participation is different from the commonly used individual rationality constraint. To further clarify, note that when non-participating users can be fully excluded (e.g., are not allocated bandwidth in the cellular network [9]), individual rationality assesses participation based on users' utilities before the introduction of a mechanism; i.e., against the Nash equilibrium. However, when the good is non-excludable, voluntary participation assesses a user's interest in the mechanism given that despite opting out, the user can

[1] We note there according to an alternative definition, see e.g. [7, Chapter 23], all public goods are assumed non-excludable; with excludable non-rivalrous goods referred to as *club goods*. However, it is more common in the literature, especially in the engineering applications' literature, to make the coarser distinction of public vs. private goods based on rivalry alone. Examples include Samuelson's seminal work on public goods [8], the definition of Mas-Colell, Whinston, and Green [6, Chapter 11.C], and [9] in the engineering applications' literature. We adopt this coarser categorization, and further distinguish based on excludability when needed.

still benefit from the externality of improved efforts by other participating users (e.g. not installing an anti-virus on her device but interacting with other computers that have done so). Hence, we use the concept of *exit equilibria* for the study of voluntary participation constraints; these equilibria will refer to the Nash equilibrium of the game between a user who unilaterally opts out of a mechanism, and the remaining users who implement the socially optimal solution in their new environment.[2]

In addition, we establish a connection between users' positions in their interdependence network (in terms of their centrality), and their effort levels at different investment profiles of these public good provision games. A graph-theoretical characterization of users' efforts in these games is of both conceptual and practical importance. On one hand, this connection allows for a comparative statics study; i.e., identifying the effects of changing the network structure on equilibrium efforts (for a given equilibrium concept). On the other hand, providing such characterization for different equilibria, on a fixed network, can serve as a tool in the problem of mechanism design.

The main contributions of this work are therefore the following. First, we identify properties of the interdependence network that guarantee the existence and uniqueness of three effort profiles (namely, Nash equilibrium, socially optimal, and exit equilibrium profiles) in public good provision games. Second, we establish a centrality-effort connection for these effort profiles. These characterizations separate the effects of incoming and outgoing dependencies, as well as the influence of paths of different length, on users' effort levels. Our selected centrality measure, the alpha-centrality, is a different (generalized) version of the measures used in the existing literature, which has mainly focused on characterizing Nash equilibria in public good provision problems (see Section V for more discussion). Third, this work is the first to study exit equilibria for analyzing users' actions under unilateral deviation from social welfare maximizing strategies, and to provide a graph-theoretical characterization of these exit equilibrium profiles.

The remainder of this paper is organized as follows. In Section II, we present a model for public good provision games, and formally describe the equilibrium concepts of interest. We find conditions for existence and uniqueness of these equilibria in Section III. Section IV establishes the centrality-effort connection in these games, followed by intuitive interpretation and numerical examples. We review the literature most relevant to this work in Section V, and conclude in Section VI with directions for future work.

II. Model and Preliminaries

A. Weighted effort public good provision games

Consider a set of N strategic users constituting the nodes $\mathcal{N}$ of a directed network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. Each user $i \in \mathcal{N}$ can choose to make an *investment*, or exert some *effort* $x_i \in [0, \infty)$. Denote the vector of all users' efforts by $\mathbf{x} := (x_1, x_2, \cdots, x_N)^T$.

We assume the utility of user i in $\mathcal{G}$ at a vector of efforts $\mathbf{x}$ is given by:

$$u_i(\mathbf{x}) := V_i(\sum_{j=1}^{N} a_{ij} x_j) - c_i x_i \ . \qquad (1)$$

Here, the *value* or *benefit* function $V_i : [0, \infty) \to [0, \infty)$ denotes the benefit to user i from the effort profile $\mathbf{x}$. The argument of the benefit function, $(A\mathbf{x})_i := \sum_{j=1}^{N} a_{ij} x_j$, is the *effective effort* experienced by user i, and is given by a weighted sum of some or all of the N users' efforts, with the coefficients a_{ij} determining the extent of i's dependencies on different users. We assume the *self-dependence* coefficients are normalized so that $a_{ii} = 1, \forall i$. The *interdependence* coefficients $a_{ij} \geq 0, \forall i \neq j$, are interpreted relative to 1, and determine the influence of other nodes on node i's benefits, with $a_{ij} > 0$ when $(j \to i) \in \mathcal{E}$.

Finally, for each user i, the cost of exerting effort is assumed linearly increasing in x_i, with the *unit cost (of effort)* denoted by $c_i > 0$.

We make the following assumptions on the functions $V_i(\cdot)$:

Assumption 1: The function $V_i(\cdot)$ is twice differentiable, strictly increasing, and strictly concave, $\forall i$. In addition, $V_i'(0) > c_i, \forall i$.

In addition to allowing mathematical tractability, these assumptions entail the following intuition. By the monotonicity assumption, users' benefits increase as more effort is exerted. However, the concavity assumption reflects diminishing marginal benefits: while initial exertion of effort increases utilities more considerably, the marginal benefit from additional effort is overall decreasing. The last requirement, $V_i'(0) > c_i$, ensures that all users will benefit from experiencing a non-zero effective effort.

Notation: Let $A = [a_{ij}]$ denote the *dependence* matrix. We further derive the *i-removed dependence* matrix A_{-i} from A by setting the entries in the row and column corresponding to i (except for the diagonal element) to zero. Formally, set $[A_{-i}]_{ik} = [A_{-i}]_{ki} = 0, \forall k \neq i$. All other entries remain unchanged, i.e., $[A_{-i}]_{jk} = a_{jk}, \forall j \neq i, k \neq i$, and $[A_{-i}]_{ii} = a_{ii}$. These matrices are illustrated below:

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix},$$

$$A_{-i} := \begin{pmatrix} a_{11} & \cdots & a_{1(i-1)} & 0 & a_{1(i+1)} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(i-1)} & 0 & a_{(i-1)(i+1)} & \cdots & a_{(i-1)N} \\ 0 & \cdots & 0 & a_{ii} & 0 & \cdots & 0 \\ a_{(i+1)1} & \cdots & a_{(i+1)(i-1)} & 0 & a_{(i+1)(i+1)} & \cdots & a_{(i+1)N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{N(i-1)} & 0 & a_{N(i+1)} & \cdots & a_{NN} \end{pmatrix}.$$

We let $B = A - I$ be the *adjacency* or *interdependence* matrix, and define the *i-removed interdependence* matrix $B_{-i} = A_{-i} - I$.

[2] We note that similar exit equilibrium concepts can be defined for coalitions and group deviations. The study of such equilibria is a direction for future work.

TABLE I

SUMMARY OF NOTATION

Symbol	Description
A	Dependence matrix
A_{-i}	i-removed dependence matrix
B	Adjacency/interdependence matrix
B_{-i}	i-removed interdependence matrix
$\mathbf{c}$	Vector of unit effort costs
$\mathbf{V}$	Vector of benefit functions
$\nabla\mathbf{V}$	1^{st} derivatives of benefit functions
$\nabla\mathbf{V}^{-1}$	Inverse of 1^{st} derivatives of benefit functions

We use $\mathbf{c} = [c_1, \cdots, c_N]^T$ and $\mathbf{V} = [V_1(\cdot), \cdots, V_N(\cdot)]^T$ to denote the vector of unit effort costs and benefit functions, respectively. In addition, denote the vector of first derivatives of the benefit functions by $\nabla\mathbf{V} = [V_i'(\cdot)] = [\partial V_i(x)/\partial x]$, and the vector of the inverse of these derivatives by $\nabla\mathbf{V}^{-1} = [(V_i')^{-1}(\cdot)]$. The notation is summarized in Table I.

We interpret $\mathbf{V}(\mathbf{x})$ as the vector of functions $\mathbf{V}$ evaluated element-wise at the vector of variables $\mathbf{x}$, i.e, its i^{th} entry is $V_i(x_i)$.

B. Equilibria in weighted effort games

The set of users $\mathcal{N}$, their actions $x_i \geq 0$, and their utility functions $u_i(\mathbf{x})$ given in (1), constitute a weighted effort public good provision game. Our goal in this paper is to identify properties of the interdependence graph that ensure the existence and uniqueness of equilibria in these games, and graph theoretical interpretations of users' actions in different effort profiles. In particular, we are interested in the study of three effort profiles:

a) **Nash equilibrium $\tilde{\mathbf{x}}$:** these are the effort levels realized at the status quo of the full information game, in which strategic users simultaneously choose their effort levels to maximize their payoff given others' actions. At the Nash equilibrium, no user has an incentive to unilaterally deviate from her action. Formally:

$$\tilde{x}_i = \arg\max_{x\geq 0} \quad u_i(x, \tilde{\mathbf{x}}_{-i}) \ , \forall i. \tag{2}$$

b) **Socially optimal solution $\mathbf{x}^*$:** this is the effort profile maximizing the sum of all users' utilities. Formally,

$$\mathbf{x}^* = \arg\max_{\mathbf{x}\succeq 0} \sum_{i=1}^{N} u_i(\mathbf{x}) \ . \tag{3}$$

The implementation of these effort profiles can be incentivized by using appropriate tax-based mechanisms [10], such as the Pivotal (VCG) [11] or Externality [12] mechanisms.

c) **Exit equilibria $\hat{\mathbf{x}}^{\mathbf{i}}, \forall i$:** the participation of users in a mechanism that incentivizes the socially optimal solution to a public good provision game is affected by their outside options. In particular, as mentioned in Section I, when the public good is non-excludable, a user can opt out from the mechanism while still benefiting from the externality of other users' (possibly reduced) efforts, and best-responding with the choice of an effort level accordingly. To study the effort profiles realized as a result of such unilateral deviations, we propose the concept of *exit equilibrium*. Formally, the exit equilibrium $\hat{\mathbf{x}}^{\mathbf{i}}$ is a Nash equilibrium of a simultaneous move game between the outlier i and the remaining participating users (who implement the socially optimal solution for their $N-1$ user network):

$$\begin{aligned} \hat{\mathbf{x}}^i_{-i} &= \arg\max_{\mathbf{x}_{-i}\succeq 0} \quad \sum_{j\neq i} u_j(\mathbf{x}_{-i}, \hat{x}^i_i) \ , \\ \hat{x}^i_i &= \arg\max_{x_i\geq 0} \quad u_i(\hat{\mathbf{x}}^i_{-i}, x_i) \ . \end{aligned} \tag{4}$$

We now proceed to the study of existence and uniqueness of the aforementioned equilibrium concepts in public good provision games.

III. Existence and Uniqueness of Effort Profiles

In this section, we first show that the optimality conditions for solving the optimization problems of Section II-B can be re-written as Linear Complementarity Problems (LCP) [13]. A general LCP is determined by a vector-matrix pair $(\mathbf{b}, A)$, and is given by:

$$\mathbf{y} = A\mathbf{x} + \mathbf{b}, \ \mathbf{y}^T\mathbf{x} = 0, \quad \mathbf{x}, \mathbf{y} \succeq 0 \ .$$

Linear complementarity problems were first proposed as a method to unify the study of linear and quadratic programming, and have found applications in the study of market equilibrium, finding optimal stopping times in Markov chains, and developing efficient algorithms for solving nonlinear programming problems [13], [14]. By finding an equivalent formulation, we will leverage results from the LCP literature to establish conditions for existence and uniqueness of effort profiles in our public good provision games.

A. The LCP Formulation

First, consider the Nash equilibrium profile found through the system of equations (2). Let $\tilde{b}_i$ denote the aggregate effort level at which a user i's marginal benefit is equal to her marginal cost; i.e., $V_i'(\tilde{b}_i) = c_i$. Then, a Nash equilibrium profile $\mathbf{x}$ should satisfy:

$$\begin{aligned} (A\mathbf{x})_i &= \tilde{b}_i \quad \text{if } x_i > 0 \ , \\ (A\mathbf{x})_i &\geq \tilde{b}_i \quad \text{if } x_i = 0 \ . \end{aligned}$$

The above can be written as the following LCP:

$$\begin{aligned} &\mathbf{y} = A\mathbf{x} - \tilde{\mathbf{b}} \\ &\mathbf{y}^T\mathbf{x} = 0 \\ &\mathbf{x} \succeq \mathbf{0} \ , \mathbf{y} \succeq \mathbf{0} \end{aligned} \tag{5}$$

Any profile $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ satisfying (5) is a Nash equilibrium of the game defined in Section II. Note that the variables y_i determine the slack for users who do not exert effort.

We next consider the socially optimal investment profile satisfying (3). A solution $\mathbf{x}$ should satisfy:

$$\begin{aligned} \sum_{j=1}^{N} a_{ji} V_j'((A\mathbf{x})_j) &= c_i, \quad \text{if } x_i > 0 \ , \\ \sum_{j=1}^{N} a_{ji} V_j'((A\mathbf{x})_j) &\leq c_i, \quad \text{if } x_i = 0 \ . \end{aligned}$$

Define the efforts b_i^* as those satisfying $V_i'(b_i^*) = ((A^T)^{-1}\mathbf{c})_i$. Then, the above optimality conditions are equivalent to:

$$(A\mathbf{x})_i = b_i^*, \quad \text{if } x_i > 0 \ ,$$
$$(A\mathbf{x})_i \geq b_i^*, \quad \text{if } x_i = 0 \ .$$

The above can be written as the following LCP:

$$\begin{aligned} &\mathbf{y} = A\mathbf{x} - \mathbf{b}^* \\ &\mathbf{y}^T\mathbf{x} = 0 \\ &\mathbf{x} \succeq \mathbf{0} \ , \mathbf{y} \succeq \mathbf{0} \end{aligned} \tag{6}$$

Any profile $(\mathbf{x}^*, \mathbf{y}^*)$ satisfying (6) is a socially optimal solution of the game defined in Section II. Note that this LCP has the same matrix A as that in (5), but a different vector $\mathbf{b}$.

Similarly, to find an exit equilibrium $\mathbf{x}^i$ of the game based on (4), in which user i has unilateraly opted out of a given mechanism, we have to find a solution to the following system of equations:

$$\sum_{j\neq i} a_{jk} V_j'((A\mathbf{x}^i)_j) = c_k, \quad \text{for } k \neq i, \text{if } x_k^i > 0 \ ,$$
$$\sum_{j\neq i} a_{jk} V_j'((A\mathbf{x}^i)_j) \leq c_k, \quad \text{for } k \neq i, \text{if } x_k^i = 0 \ ,$$
$$V_i'((A\mathbf{x}^i)_i) = c_i, \quad \text{if } x_i^i > 0 \ ,$$
$$V_i'((A\mathbf{x}^i)_i) \leq c_i, \quad \text{if } x_i^i = 0 \ .$$

Defining the efforts $\hat{b}_k^i$ as those satisfying $V_k'(\hat{b}_k^i) = ((A_{-i}^T)^{-1}\mathbf{c})_k$, and following similar steps, the above can be written as the following LCP problem:

$$\begin{aligned} &\mathbf{y} = A\mathbf{x}^i - \hat{\mathbf{b}}^{\mathbf{i}} \\ &\mathbf{y}^T\mathbf{x}^i = 0 \\ &\mathbf{x}^i \succeq \mathbf{0} \ , \mathbf{y} \succeq \mathbf{0} \end{aligned} \tag{7}$$

Any profile $(\hat{\mathbf{x}}^i, \hat{\mathbf{y}}^i)$ satisfying (7) is an exit equilibrium solution under user i's unilateral deviation for the game defined in Section II.

B. Existence and Uniqueness

Based on the LCP formulations derived in the previous section, the conditions for the existence and uniqueness of the equilibria of public good provision games will be equivalent to those of the corresponding LCP problems.

Consider a general LCP $(\mathbf{b}, A)$ given by:

$$\mathbf{y} = A\mathbf{x} + \mathbf{b}, \ \mathbf{y}^T\mathbf{x} = 0, \quad \mathbf{x}, \mathbf{y} \succeq 0 \ .$$

We further use the following definitions:

Definition 1: A square matrix A is a *P-matrix* if all its principal minors (i.e., the determinant of smaller square sub-matrices obtained from A by removing one or more of its rows and columns) are positive.

Definition 2: A matrix A is *strictly diagonally dominant* if $\sum_{j\neq i} |a_{ij}| < |a_{ii}|, \ \forall i$.

We will use the following existence and uniqueness result from the LCP literature, see e.g. [13, Theorem 3.3.7].

TABLE II
CLOSED FORM OF NE, SO, AND EEs

Profile	Closed form
Nash equilibrium	$\tilde{\mathbf{x}} = A^{-1}\nabla\mathbf{V}^{-1}(\mathbf{c})$
Socially optimal	$\mathbf{x}^* = A^{-1}\nabla\mathbf{V}^{-1}((A^T)^{-1}\mathbf{c})$
Exit equilibrium	$\hat{\mathbf{x}}^i = A^{-1}\nabla\mathbf{V}^{-1}((A_{-i}^T)^{-1}\mathbf{c})$

Theorem 1: The LCP $(\mathbf{b}, A)$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^N$ if and only if $A \in \mathbb{R}^N \times \mathbb{R}^N$ is a P-matrix.

The following theorem establishes conditions for existence and uniqueness of equilibria of the public good provision games studied in this paper.

Theorem 2: If the dependence matrix A is strictly diagonally dominant, then the public good provision game defined in Section II has unique Nash equilibrium, socially optimal, and exit equilibria.

Proof: We first show that if A is strictly diagonally dominant, then it is a P-matrix. This is because by the Gershgorin circle theorem, for a strictly diagonally dominant matrix with positive diagonal elements, all real eigenvalues are positive. Following a similar argument, all real eigenvalues of all sub-matrices of A are also positive. Since the determinant of a matrix is the products of its eigenvalues, and as for real matrices, the complex eigenvalues appear in pairs with their conjugate eigenvalues, it follows that A, as well as all its square sub-matrices, have positive determinants. Therefore, A is a P-matrix. Using Theorem 1 we conclude that all the LCPs derived in (5), (6), and (7), have unique solutions, regardless of the value of $\mathbf{b}$. ■

C. Closed form of interior NE, SO, and EEs

In the remainder of the paper, we will focus on game environments for which at any equilibrium, all entries of the effort profile are strictly positive.[3] Intuitively, this requirement implies that any user, regardless of externalities, will still benefit from exerting some effort herself. Technically, this assumption allows us to derive closed forms for the interior solutions of the corresponding optimization problems. Table II provides the closed form of interior solutions of the three aforementioned effort profiles. The derivation follows by substituting for $\tilde{\mathbf{b}}, \mathbf{b}^*, \hat{\mathbf{b}}^{\mathbf{i}}$ and setting the corresponding slack variables $\mathbf{y}$ to zero in the LCP formulations of Section III-A.

IV. The Centrality-Effort Connection

A. An overview of centrality measures

Several measures of node centrality have been proposed throughout the graph theory and network analysis literature. A commonly used measure, degree centrality, declares the node with the highest number of links as the most central. In contrast to degree centrality and other similar measure, another class of measures propose accounting for

[3] Intuitively, it is sufficient to have small enough interdependence coefficients $a_{ij}, \forall j \neq i$, or to focus on sufficiently sparse networks. Identifying conditions that guarantee the existence of interior effort profiles remains as a direction for future work.

the importance of the connections rather than the number of connections alone. We are interested in the latter family of measures, and specifically, the *alpha-centrality* measure.

The alpha-centrality measure was introduced by Bonacich and Lloyd in [16], primarily as a centrality measure that is applicable to networks with asymmetric relations. It is defined as follows. Let B be the adjacency matrix of a network, with b_{ij} determining the dependence of node i on j. Assume each node's centrality, x_i, is influenced by the centrality of her neighbors; this leads to an eigenvector centrality measure, $\mathbf{x} = B\mathbf{x}$. Alpha-centrality generalizes this measure by allowing nodes to also enjoy an exogenous source of centrality $\mathbf{e}$. As a result, the vector of alpha-centralities satisfies:

$$\mathbf{x} = \alpha B\mathbf{x} + \mathbf{e} \ .$$

Here, α determines the tradeoff between the importance of the endogenous and exogenous centrality factors. The vector of alpha-centralities is therefore given by:

$$c_{\text{alpha}}(B, \alpha, \mathbf{e}) = (I - \alpha B)^{-1}\mathbf{e} \ . \tag{8}$$

The alpha-centrality measure is closely related to the centrality measure proposed in the seminal work of Katz [17]. Katz' measure defines a weighted sum of powers of the adjacency matrix B as an indicator of nodes' importance; intuitively, longer paths are weighed differently (often less favorably) in determining nodes' centralities. Formally, the Katz' measure is given by:

$$c_{\text{katz}}(B, \alpha) = (\sum_{i=1}^{\infty} \alpha^i B^i)\mathbf{1} \ ,$$

where α is an attenuation factor. In particular, if $\alpha < \frac{1}{|\lambda_1(B)|}$ (where $\lambda_1(B)$ is the largest eigenvalue of B), this infinite sum converges to $(I - \alpha B)^{-1} - I$. Therefore:

$$(\sum_{i=1}^{\infty} \alpha^i B^i)\mathbf{e} = (-I + \sum_{i=0}^{\infty} \alpha^i B^i)\mathbf{e} = (-I + (I - \alpha B)^{-1})\mathbf{e} \ . \tag{9}$$

Comparing (8) and (9), we conclude that the parameter α in alpha-centrality measures can be similarly interpreted as a weight assigned to the paths of different length in determining the effect of endogenous centralities on the overall centrality of a node.[4]

[4] Alpha centralities are also similar to the measure introduced earlier by Bonacich in his seminal work [18]. Formally, Bonacich's centrality is defined as $c_{\text{bonacich}}(R, \beta, \alpha) = \beta(I - \alpha R)^{-1}R\mathbf{1}$. Here, R is a symmetric matrix of relationships, with main diagonal elements equal to zero. The parameter β only affects the length of the final measures, and has no network interpretation. The parameter α on the other hand can be positive or negative, and determines the extent and direction of influences. On symmetric matrices, this measure is essentially equivalent to Katz's measure as well; in fact, $c_{\text{katz}}(R, \alpha) = \sum_{i=1}^{\infty} \alpha^i R^i \mathbf{1} = \alpha c_{\text{bonacich}}(R, \alpha, 1)$.

To summarize, taking the three measures on a symmetric matrix A, and setting $\mathbf{e} = \mathbf{1}$ for the alpha-centralities, we have:

$$c_{\text{alpha}}(A, \alpha, \mathbf{1}) = 1 + \alpha c_{\text{bonacich}}(A, \alpha, 1) = 1 + c_{\text{katz}}(A, \alpha) \ .$$

Therefore, in essence, alpha-centrality generalizes Bonacich and Katz centralities, allowing for vectors of exogenous status $\mathbf{e}$.

B. *Effort profiles as alpha-centralities*

Using alpha-centrality, we can describe the interior Nash equilibrium, socially optimal, and exit equilibrium profiles of public good provision games. This can be done by replacing $A = I + B$ in Table II, and comparing the expressions with (8). The results are summarized in Table III.

TABLE III

NE, SO, AND EEs AS NODE CENTRALITIES

Profile	alpha-centralities
$\tilde{\mathbf{x}}$	$c_{\text{alpha}}\left(B, -1, \nabla\mathbf{V}^{-1}(\mathbf{c})\right)$
$\mathbf{x}^*$	$c_{\text{alpha}}\left(B, -1, \nabla\mathbf{V}^{-1}\left(c_{\text{alpha}}(B^T, -1, \mathbf{c})\right)\right)$
$\hat{\mathbf{x}}^i$	$c_{\text{alpha}}\left(B, -1, \nabla\mathbf{V}^{-1}\left(c_{\text{alpha}}(B^T_{-i}, -1, \mathbf{c})\right)\right)$

These centrality-based formulations of effort profiles allow for a better understanding of users' actions in different effort profiles. In particular, we make the following observations.

1) A comparison across equilibria: First, note that the outer centrality measures – determining users' efforts in the three equilibria – differ only in the vector of exogenous status. These vectors can be interpreted as efforts at which users' marginal costs equal their marginal benefits. This is easily observable in $\tilde{\mathbf{e}} := \nabla\mathbf{V}^{-1}(\mathbf{c})$ at the Nash equilibrium.

For the socially optimal solution on the other hand, $\mathbf{e}^* := \nabla\mathbf{V}^{-1}\left(c_{\text{alpha}}(B^T, -1, \mathbf{c})\right)$ indicates that users' marginal costs now equal a weighted version of all users' costs. This is to be expected, as in a socially optimal solution, users will be accounting for the externality of their actions when making optimal effort decisions, hence a modified cost perception.

Finally, for user i's exit equilibrium, we get $\hat{\mathbf{e}}^{\mathbf{i}} := \nabla\mathbf{V}^{-1}\left(c_{\text{alpha}}(B^T_{-i}, -1, \mathbf{c})\right)$; i.e., the equilibrium costs are evaluated on the i-removed adjacency matrix B_{-i}, indicating that user i's interest no longer influences the participating nodes' cost perceptions.

Based on these observations, we will henceforth refer to the argument z of the $\nabla\mathbf{V}^{-1}(z)$ functions as users' *perceived costs*. Note that the vectors of perceived costs also appear as the vectors $\mathbf{b}$ in the corresponding LCP formulations of Section III-A.

2) Effect of incoming and outgoing dependencies: Next, we observe that the perceived costs at the socially optimal and exit equilibria are the alpha-centralities of nodes, calculated on the *transpose* of the appropriate adjacency matrices. Intuitively, this implies that a user j has to consider her *outgoing* influences b_{kj} (for participating users only) in determining perceived costs; i.e., other users' dependence on her.

On the other hand, note that the outer alpha-centrality measures are all calculated on the *original* adjacency matrix B, regardless of the equilibrium concept. This means that a user j will be considering all *incoming* influences b_{jk} – user k's participation and the solution implemented notwithstanding – as all other users can provide positive externality to user j regardless.

3) The interpretation of α: Finally, we note that all alpha parameters in these measures are $\alpha = -1$.[5] Considering alpha's interpretation as a weight for paths of different length, this implies that paths of odd length are weighed positively, while those of even length are weighed negatively. This is consistent with users' actions in public good provision games; e.g., a user will decrease her effort in response to an immediate neighbor's increased effort; however, she will increase her effort in response to a neighbor two-hops away increasing effort. A similar interpretation applies to the alpha-centralities determining perceived costs.

C. Numerical Examples

In this section, we illustrate the results of Table III and the ensuing intuition through numerical examples.

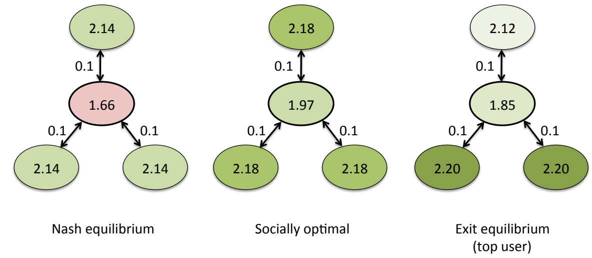
1) Comparison across equilibria on a given network: We start by considering the 4-node star network of Fig. 1. We index the center as user 1, and the top user as user 4. All users are assumed to have equal benefit functions $V_i(x) = 1 - exp(-x), \forall i$, and equal unit costs of effort $c_i = 0.1, \forall i$. We further assume a symmetric 0.1 dependence between the center and the leaf nodes. We use the formulas in Table III to find the different effort profiles in this network, and make the following observations.

First, comparing users' efforts at any given profile (the numbers inside the nodes) illustrates how users' efforts are, as expected, consistent with the positive externalities available to them as a result of their centrality in the network. In particular, by being situated in a more central position in the network, user 1 can benefit from high positive externalities. This is due to the availability of 3 paths of odd length, which given $\alpha = -1$, decrease user 1's alpha-centrality (the outer alpha-centrality in Table III). Hence, compared to other users, user 1 exerts lower effort at any of the effort profiles.

Next, we take a closer look at the vectors of perceived costs. First note that the vector of perceived costs at the socially optimal solution is modified according to nodes' alpha-centralities. In particular, as user 1 again has a lower alpha-centrality (the inner alpha-centrality in Table III), it has a lower perceived cost compared to other nodes. A lower perceived cost results in a higher exogenous status (due to the concavity of the benefit functions), which in turn translates to an increased outer alpha-centrality for the user. This effect is reflected in the considerable improvement of user 1's effort from the Nash equilibrium to the socially optimal profile.

Note also that at the exit equilibrium, the perceived cost of the outlier (last entry of $\hat{c}^4$) remains unchanged, as this user only perceives her own cost after exiting. The perceived costs by the other users at this exit equilibrium are also higher than that of the socially optimal solution, as the participating users no longer account for the effect of their choices on the welfare of the deviating user. Accordingly, the effort levels of the participating users have decreased at the exit equilibrium.

[5]Given $\alpha = -1$, the condition $\alpha < \frac{1}{|\lambda_1(B)|}$ holds for all adjacency matrices B. Therefore, the alpha-centralities can be interpreted as the limit of a weighted sum of powers of the adjacency matrix, and the interpretation of α as a weight on paths of different length in applicable (see Section IV-A).



$$\tilde{c} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} \quad c^* = \begin{bmatrix} 0.072 \\ 0.093 \\ 0.093 \\ 0.093 \end{bmatrix} \quad \hat{c}^i = \begin{bmatrix} 0.082 \\ 0.092 \\ 0.092 \\ 0.1 \end{bmatrix}$$

Fig. 1. Effort profiles (numbers inside nodes) and vectors of perceived costs across different effort profiles on a given network.

2) Effects of incoming dependencies: We next consider the 4-node network illustrated in Fig. 2. We again index the center user as 1, and assume identical benefit functions $V_i(x) = 1 - exp(-x), \forall i$, and equal unit costs of effort $c_i = 0.1, \forall i$ for all users. The network of the top row assumes a 0.1 dependence of leaf nodes on the center, while the network of the bottom row assumes an increased 0.3 dependence on user 1. The Nash equilibrium and socially optimal profiles, as well as the vector of equivalent costs at the socially optimal solution, are illustrated in Fig. 2.

We immediately observe that the cost perceived by user 1 at the socially optimal solution is considerably lower in the bottom network. This is due to the fact that when the weight of incoming links to the center node increases, the inner alpha-centrality $c_{\text{alpha}}(B^T, -1, \mathbf{c})$ decreases (note that this inverse relation is a consequence of $\alpha = -1$). Consequently, to maximize welfare, user 1 is required to exert much higher effort in the bottom scenario as compared to the top network.

We conclude that, as the inner alpha-centralities of Table III suggest, users' cost perceptions, and consequently their optimal effort levels, are highly affected by their incoming dependencies. It is also interesting to note that in the bottom network, the dependence of users on the center (and hence the center's efficiency in providing the good) is so high that the leaves are required to decrease efforts to reach a welfare maximizing solution.

V. Related work

In the economic applications' literature, the work in [3], [19], [1] establish connections between the Nash equilibrium efforts and the interdependence graph in public good provision environments. In [3], Bramoulle and Kranton introduce a network model of public goods (a special case of the model considered herein), and study different features of its Nash equilibria. Specifically, it is shown that these games always have a *specialized* Nash equilibrium – one in which users

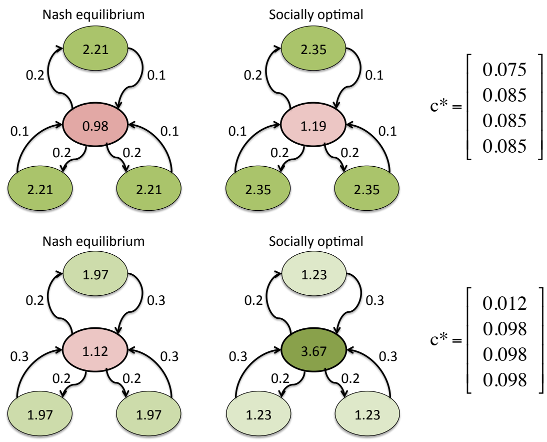


Fig. 2. Effects of incoming and outgoing dependencies.

are either specialists exerting full effort, or free-riders – and that such equilibria correspond to maximal independent sets of the network graph. In contrast, our focus in this paper is on a graph-theoretical characterization of non-specialized, or distributed, Nash equilibria (as well as other distributed effort profiles).

Similarly, Ballester et al. [19] study the (distributed) Nash equilibrium of a linear quadratic interdependence model, and relate the equilibrium effort levels to the nodes' Bonacich centralities in a suitably defined matrix of local complementarities. The work in [1] generalizes these results, by studying existence, uniqueness, and closed form of the Nash equilibrium in a broader class of games for which best-responses are linear in other players' actions. They further characterize the Nash equilibrium in terms of an expression containing a generalized form of the Katz centrality measure. In addition to studying a different interdependence model and centrality measure, our work differs in that it focuses not only on the Nash equilibrium, but also on other effort profiles.

The current paper is also closely related to the work of Elliott and Golub in [2], which focuses on implementation of Pareto efficient public good outcomes, rather than the Nash equilibria on a given network. The authors define a *benefits matrix* for any given network graph; an entry B_{ij} of the matrix is the marginal rate at which i's effort can be substituted by the externality of j's action. The main result of the paper states that efforts at a Lindahl outcome can be characterized using the eigenvector centrality of this benefits matrix. The current paper differs from [2] in the following aspects. In terms of modeling, although [2] does not require users' preferences to be separable in costs and benefits, a user's action is assumed to be strictly costly for the user herself; whereas our model allows users to benefit from their own effort. More importantly, the focus of our work is on socially optimal solutions, and unilateral deviations resulting in exit equilibria (required to analyze voluntary participation), while [2] mainly focuses on Pareto efficiency and individual rationality, with the study of voluntary participation suggested as future work.

In the context of security games, the weighted effort game model studied herein is similar to the games studied in [20], [21], [15], [22]. Our model is a generalization of the total effort model proposed by Varian in his seminal work [20], and is similar to the effective investment model in [21], and the linear influence network game in [15], [22]. The effective investment model in [21] has been considered to determine a bound on the price of anarchy gap, i.e. the gap between the socially optimal and Nash equilibrium efforts. The linear influence models in [15], [22] have been proposed to study properties of the interdependence matrix affecting the existence and uniqueness of the Nash equilibrium, as well as an iterative algorithm for converging to this equilibrium (however, no graph theoretical interpretation of the Nash equilibrium has been given). Our results on the existence and uniqueness of Nash equilibrium in Section III are consistent with those of [15], [22]. Our work on this model therefore closes the gap in this literature by studying the effect of the interdependence network on the existence and uniqueness of socially optimal and exit equilibrium profiles, and further provides graph-theoretical interpretations of all three effort profiles.

VI. Conclusion

We have identified conditions on the dependence matrix that guarantee the existence and uniqueness of different effort profiles in weighted effort public good provision games. We further established a connection between users' positions in their interdependence network (in terms of alpha-centralities), and their actions in different interior effort profiles. These characterizations can be useful in pursuing several directions. First, the results can be used in designing tax/subsidy policies. Specifically, the unit costs of effort could be appropriately taxed/subsidized to modify the vector of perceived costs, and consequently, arrive at a desired effort profile.

Second, these characterizations can be used in studying the performance of specific incentive mechanisms. For example, it can be shown [10] that the use of a Pivotal (VCG) mechanism in the provision of a non-excludable public good can introduce taxes that incentivize the socially optimal effort profile and guarantee voluntary participation, but may result in a budget deficit in some instances of the game. The current work can be continued to study the connection between the network structure and game instances in which a deficit is incurred, as well as the size of this deficit.

Finally, extending this framework to study coalitions and group deviations from incentive mechanisms is an important direction of future work.

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