Safe control using high-order measurement robust control barrier functions

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Abstract— **We study the problem of providing safety guarantees for dynamic systems of high relative degree in the presence of state measurement errors. To this end, we propose High-Order Measurement Robust Control Barrier Functions (HO-MR-CBFs), an extension of the recently proposed Measurement Robust Control Barrier Functions. We begin by formally defining HO-MR-CBF, and identify conditions under which the proposed HO-MR-CBF can render the system's safe set forward invariant. In addition, we provide bounds on the state measurement errors for which the optimization problem for identifying the corresponding safe controllers is feasible for all states within the safe set and given restricted control inputs. We demonstrate the proposed approach through numerical experiments on a collision avoidance scenario in presence of measurement noise using a nonlinear kinematic model of a wheeled robot. We show that using our proposed control method, the robot, who has access to only biased state estimates, will be successful in avoiding the obstacle.**

I. INTRODUCTION

Automated Cyber-Physical Systems (CPS) increasingly perform safety-critical functions in different applications, such as transportation systems, energy delivery, and health-care. As such, designing *safe* controllers for such systems has been a topic of growing importance and interest.

Among existing approaches to safe control, Control Barrier Functions have gained increasing popularity [1], [2] due to their ability to assure safety with strong analytical guarantees using tools from *set invariance* [3]. Along with Control Lyapunov Functions (CLFs), they have been shown to be capable of providing real-time safe control by formulating and solving a Quadratic Program (QP) [4]. They have been successfully implemented in several applications including bipedal robots control [5], [6], automotive safety [1], and machine learning applications [7]–[9]. CBF have also been extended through several variations; examples include discrete-time CBF [5] and stochastic CBF [10]. The most closely related variation to our paper is the High-Order CBF [11], [12], designed for systems with *high relative degree* (formally defined in Section II-C). In particular, many CBF approaches are only applicable when the constraint has degree one relative to the dynamics of the system; HOCBF allow for safe control when this assumption does not hold.

Despite their ubiquity, the vast majority of the existing works on CBF (and their variants) assume that the controller has access to *perfect state information*. However, this is often not the case in practice, especially given limitations in sensing technologies, as well as the rise in cyber attacks on automated CPS.

In particular, CPS typically consist of a large network of physical actuators, sensors, and controllers, creating a vast attack surface, including for launching physical attacks (e.g., false sensor signals) than can bypass traditional security mechanisms such as authentication and encryption [13]–[15]. At the lowest level, these attacks manifest themselves as false input and/or output signals. While outlier and anomaly rejection algorithms such as the χ^2-detector are successful in raising an alarm at high noise levels, an intelligent attacker can bypass these algorithms to induce maximum estimation error without raising an alarm [16]. This is done through a *stealthy attack* signal that is masked within the sensor noise [17]. It is therefore essential to design a control architecture that is robust against state measurement errors, and thus ensures safe functioning irrespective of whether the presence of an attack is detected (passive safety).

To address this shortcoming of existing CBF, a number of recent works have developed extensions to account for potential state measurement errors. Control Barrier Functions in the presence of measurement noise have been proposed from a stochastic perspective in [18]. Measurement Robust CBFs (MR-CBF) were proposed in [19] mainly motivated by ensuring safety in the presence of uncertainty in measurements in vision-based systems. This MR-CBF formulation was further augmented with backup sets and successfully showcased on an autonomous Segway in [20]. However, all these recently proposed measurement robust CBF are only applicable in systems with relative degree one. Our work takes inspiration form existing works [11], [19], [20], and aims to integrate concepts from measurement robustness with high-order CBFs to ensure safe control in systems with high relative degree in the presence of sensor noise.

In particular, we use the concept of Control Barrier Functions to build safety filters capable of generating control signals which ensure that the state trajectory is safe despite state measurement uncertainties. We provide such safety guarantees using *High-Order Measurement Robust Control Barrier Functions (HO-MR-CBF)*, and show how to attain safe controllers for both unconstrained systems and those with restricted (bounded) controls. Specifically, we make the following contributions:

- We generalize Measurement Robust Control Barrier Functions [19], [20] by proposing High-Order Measurement Robust Control Barrier Functions (HO-MR-CBF) which allow for safe control using CBFs with high-

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relative degree in presence of state measurement errors.
- We provide upper bounds on the measurement errors, for both restricted and unrestricted control inputs, under which the resulting optimization problem for identifying safe controllers is feasible.
- We showcase our approach on an obstacle avoidance problem using a nonlinear kinematic robot model and sensor noise. We show that both existing measurement robust CBF and high-order CBF fail in this task, while our proposed HO-MR-CBF successfully identifies a safe controller and avoids the obstacle.

Paper organization. Some preliminaries on Control Barrier Functions (CBFs) and relevant variations are presented in Section II. The HO-MR-CBF is proposed in Section III, where we first provide the conditions under which the safe set can be rendered forward invariant in presence of measurement uncertainty using this function (Section III-A), followed by the bounds on the error in the state estimation to guarantee safety of all states in the safe-set using a HO-MR-CBF (Section III-B). Section IV showcases the proposed methodology using a simulation of a wheeled robot performing obstacle avoidance in presence of state measurement noise. Section V provides a conclusion and future directions.

II. Preliminaries

A. Safe Sets and Control Barrier Functions

Consider the continuous time, control affine system given by:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) + g(\mathbf{x}(t))\mathbf{u}(t) \tag{1}$$

where $\mathbf{x} \in \mathbb{X} \subseteq \mathbb{R}^n$ denotes the system state, $\mathbf{u} \in \mathbb{U} \subset \mathbb{R}^m$ is the control input, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{nxm}$ are locally Lipshitz functions.

We consider a notion of safety defined using a *safe set* $\mathcal{S}$ of system states $\mathbf{x}$, formalized as the 0-superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$:

$$\mathcal{S} := \{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \geq 0\}. \tag{2}$$

Denote the boundary and interior of $\mathcal{S}$ by $\partial S := \{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) = 0\}$ and $\text{Int}(S) := \{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) > 0\}$, respectively.

System (1) is said to be *safe* with respect to a set $\mathcal{S}$ if $\mathcal{S}$ is *forward invariant*, as defined below.

Definition 1 (Forward invariance): The set $\mathcal{S} \subseteq \mathbb{R}^n$ is a forward invariant set if $\mathbf{x}(t) \in \mathcal{S}$, $\forall t \in [0, t_{max})$ when $x(0) \in \mathcal{S}$. Here, $[0, t_{max})$ is the interval of existence of the solution $\mathbf{x}(t)$ for the initial condition $\mathbf{x}(0)$.

To assure the safety of system (1), we need to identify *safe* control inputs under which the set $\mathcal{S}$ would be rendered forward invariant. We do so by stating the conditions on $h(\mathbf{x})$, the function defining the safe set $\mathcal{S}$, that would allow us to find such control inputs. Before stating these conditions, we first introduce *class $\mathcal{K}$ functions*, a mathematical tool used in comparing nonlinear functions.

Definition 2 (Class $\mathcal{K}$ function [21]): A continuous function $\alpha : [0, a] \to [0, \infty]$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

With this, the definition of a *Control Barrier Function (CBF)* is provided below:

Definition 3 (Control Barrier Function [22], [23]): Given a set $\mathcal{S}$ defined by (2), $h(\mathbf{x})$ is a Control Barrier Function (CBF) if there exists a class $\mathcal{K}$ function α such that,

$$L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u} + \alpha(h(\mathbf{x})) \geq 0, \ \forall \mathbf{x} \in \mathcal{S}. \tag{3}$$

Here, $L_f h(\mathbf{x})$ denotes the Lie derivative of h along f[1].Given a CBF $h(\mathbf{x})$ on set $\mathcal{S}$ defined as (2), any Lipschitz continuous controller $\mathbf{u} \in \mathbb{K}_{\text{CBF}}(\mathbf{x})$ renders the set $\mathcal{S}$ forward invaraint for system (1) where

$$\mathbb{K}_{\text{CBF}}(\mathbf{x}) := \{\mathbf{u} \in \mathbb{U} : L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u} + \alpha(h(\mathbf{x})) \geq 0\} \tag{4}$$

Putting these together, a family of optimal *safe* controllers can be obtained by solving the following pointwise Quadratic Program (QP):

$$\begin{aligned} K_{\text{safe}}(\mathbf{x}) &= \underset{\mathbf{u} \in \mathbb{U}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{u} - K_{\text{perf}}(\mathbf{x})\|_2 \\ \text{s.t. } & L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u} + \alpha(h(\mathbf{x})) \geq 0 \end{aligned} \tag{CBF-QP}$$

where $K_{\text{perf}}(\mathbf{x})$ is a *safety-agnostic* or potentially malicious performance controller. Following (CBF-QP), $K_{\text{safe}}(\mathbf{x})$ now acts as a safety filter whose output ensures that the system trajectory always remains within the designed safety region.

B. Measurement Robust Control Barrier Functions

The above procedure for defining a CBF for a system assumes perfect state information. In real systems, however, only measurements $\mathbf{z}(t)$ of the true system state $\mathbf{x}(t)$ are obtained (e.g., from a sensor). Formally, assume $\mathbf{z}(t) = s(\mathbf{x}(t))$, were the (stochastic) function $s(\cdot)$ determines the uncertainty in measurements. Given an observation $\mathbf{z}$, an estimate of the state is obtained, using an estimation function $q(\cdot)$, as follows:

$$\hat{\mathbf{x}} := q(\mathbf{z}) = \mathbf{x} + \mathbf{e}(\mathbf{x}) \tag{5}$$

where $\mathbf{e}(\mathbf{x})$ is some unknown error function. In many cases, although $\mathbf{e}(\mathbf{x})$ is not known, the bounds on the error are known or can be obtained. This can be characterized by considering $\mathbf{e}(\mathbf{x}) \in \mathcal{E}(\mathbf{z})$ for a measurement dependent, compact set $\mathcal{E}(\mathbf{z})$, and

$$\max_{\mathbf{e} \in \mathcal{E}(\mathbf{z})} \|\mathbf{e}\|_2 \leq \epsilon(\mathbf{z}) \tag{6}$$

for some locally Lipschitz function $\epsilon : \mathbb{R}^n \to \mathbb{R}$.

To ensure safety in the presence of such measurement uncertainties, [19], [20] recently proposed *measurement robust* control barrier functions, defined below.

[1]Given a function $p(\mathbf{x})$, $L_q p(\mathbf{x}) := \nabla p(\mathbf{x}) \cdot q(\mathbf{x})$, and is called the *Lie derivative* of p along q.

Definition 4 (MR-CBF [19], [20]): For a safe set $\mathcal{S}$ defined in (2), the function $h : \mathbb{R}^n \to \mathbb{R}$ is a Measurement Robust Control Barrier Function (MR-CBF) with parameter functions $(a, b) : \mathbb{R}^n \to \mathbb{R}_+$, if there exists a class $\mathcal{K}$ function such that for all $(\mathbf{z}, \hat{\mathbf{x}}) \in \hat{v}(\mathcal{S})$:

$$\sup_{u \in \mathbb{U}} L_f h(\hat{\mathbf{x}}) + L_g h(\hat{\mathbf{x}})\mathbf{u} - (a(\mathbf{z}) + b(\mathbf{z})\|\mathbf{u}\|_2) > \alpha(h(\hat{\mathbf{x}})) \quad (7)$$

where $\hat{v}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n = (s(\mathbf{x}), q(s(\mathbf{x})))$ is the measurement-estimate function, and $\hat{v}(\mathcal{S})$ denotes the image of the safe set $\mathcal{S}$ under this function.

Assume that the functions $L_f h : \mathbb{R}^n \to \mathbb{R}$, $L_g h : \mathbb{R}^n \to \mathbb{R}$, and $\alpha \circ h : \mathbb{R}^n \to \mathbb{R}$ in (7) are Lipschitz continuous with Lipschitz constants $\bar{\mathrm{L}}_{L_f h}$, $\bar{\mathrm{L}}_{L_g h}$, and $\bar{\mathrm{L}}_{\alpha \circ h}$, respectively. Then, given such a measurement robust CBF, [19], [20] show that $\mathcal{S}$ is forward invariant under locally Lipschitz controllers taking values from

$$\begin{aligned} \mathbb{K}_{\text{MR-CBF}}(\mathbf{z}, \hat{\mathbf{x}}) := \{\mathbf{u} \in \mathbb{U} : L_f h(\hat{\mathbf{x}}) + L_g h(\hat{\mathbf{x}})\mathbf{u} \\ - (a(\mathbf{z}) + b(\mathbf{z})\|\mathbf{u}\|_2) > \alpha(h(\hat{\mathbf{x}}))\} \end{aligned} \quad (8)$$

with $a(\mathbf{z}) = \epsilon(\mathbf{z})(\bar{L}_{L_f h} + \bar{L}_{L_g h})$ and $b(\mathbf{z}) = \epsilon(\mathbf{z})(\bar{L}_{\alpha \circ h})$.

A quadratic program to find the optimal, measurement robust, safe control inputs can then be set up similar to (CBF-QP), with the constraint determined by (8).

C. High-Order Control Barrier Functions

In some systems, the system dynamics in (1) may be such that $L_g h(\mathbf{x}) = 0$; for examples, see [11] and Section IV. This will prevent the use of CBF-based methods presented above for identifying safe controllers for these systems. In particular, this would prevent the formulation of the optimization problem in (CBF-QP) discussed above, as the terms capturing the impact of control actions $\mathbf{u}$ would vanish. *High-order* control barrier functions have been proposed [11] to enable safe control in such systems, as detailed below.

Definition 5 (Relative Degree): The relative degree of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ with respect to system (1) is the number of times, m, we need to differentiate h along the dynamics of (1) until the control $\mathbf{u}$ shows up explicitly in $h^{(m)}$.

Assume the function $h(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ has a relative degree $m > 1$ (this is what is referred to as *high-order*). Define a series of functions $\psi_{0,1,\ldots,m} := \{\psi_0, \psi_1, \ldots, \psi_m\}$, with $\psi_i : \mathbb{R}^n \to \mathbb{R}, \forall i$, as follows:

$$\begin{aligned} &\psi_0(\mathbf{x}) := h(\mathbf{x}) \\ &\psi_1(\mathbf{x})) := \dot{\psi}_0(\mathbf{x}) + \alpha_1(\psi_0(\mathbf{x})) \\ &\vdots \\ &\psi_m(\mathbf{x}) := \dot{\psi}_{m-1}(\mathbf{x}) + \alpha_m(\psi_{m-1}(\mathbf{x})) \end{aligned} \quad (9)$$

where $\alpha_{1,2,\ldots,m}$ are class $\mathcal{K}$ functions. Further define a series of sets $S_{1,2,\ldots m} := \{S_1, S_2, \ldots, S_m\}$ as below:

$$\begin{aligned} &S_1 := \{\mathbf{x} \in \mathbb{R}^n : \psi_0(\mathbf{x}) \geq 0\} \\ &S_2 := \{\mathbf{x} \in \mathbb{R}^n : \psi_1(\mathbf{x}) \geq 0\} \\ &\vdots \\ &S_m := \{\mathbf{x} \in \mathbb{R}^n : \psi_{m-1}(\mathbf{x}) \geq 0\} \end{aligned} \quad (10)$$

In this paper, we assume that the CBF is time-invariant, which leads to the following definition:

Definition 6 (High-Order Control Barrier Function [11]): Let $\psi_{0,1,\ldots,m}$ and $S_{1,2,\ldots m}$ be defined as above. A function $h : \mathbb{R}^n \to \mathbb{R}$ is a high-order control barrier function if there exits class $\mathcal{K}$ functions $\alpha_{1,2,\ldots,m}$ such that

$$L_f^m h(\mathbf{x}) + L_g L_f^{m-1} h(\mathbf{x})\mathbf{u} + O(h(\mathbf{x})) + \alpha_m(\psi_{m-1}) \geq 0 \quad (11)$$

where $O(.) = \sum_{i=1}^{m-1} L_f^i(\alpha_{i-1} \circ \psi_{m-i-1})$.

It is shown in [11] that the set $S_1 \cap S_2 \ldots \cap S_m$ can be rendered forward invariant for system (1) if $h(\mathbf{x})$ is a high-order control barrier function that is m^{th} order differentiable.

In the remainder of the paper, we extend such high-order CBFs to be measurement robust.

III. High-Order Measurement Robust Control Barrier Functions

A. Extending MR-CBF to High-Order MR-CBF

In this section, we introduce *High-Order Measurement Robust Control Barrier Functions (HO-MR-CBF)*. We first introduce the definition, and then provide the conditions for the safety of system (1) with respect to a safe-set $\mathcal{S}$ defined by (2), given the existence of a time-invariant HO-MR-CBF $h : \mathbb{R}^n \to \mathbb{R}$.

Definition 7: Let ψ_0, ψ_1, …, ψ_m be defined by (9), and $S_1, S_2, \ldots, S_m$ be defined by (10). A function $h : \mathbb{R}^n \to \mathbb{R}$ is a High-Order Measurement Robust Control Barrier Function (HO-MR-CBF) of relative degree m for system (1) if there exist continuously differentiable class $\mathcal{K}$ functions α_1, α_2, …, α_m, and parameter functions $(a_1, a_2, \ldots a_m, b) : \mathbb{R}^n \to \mathbb{R}$ such that for all $(\mathbf{z}, \hat{\mathbf{x}}) \in \hat{v}(\mathcal{S})$:

$$\begin{aligned} L_f^m h(\hat{\mathbf{x}}) + L_g L_f^{m-1} h(\hat{\mathbf{x}})\mathbf{u} + O(h(\hat{\mathbf{x}})) + \alpha_m(\psi_{m-1}) \\ -(\sum_{i=1}^{m} a_i(\mathbf{z}) + b(\mathbf{z})\|\mathbf{u}\|_2) > 0 \ . \end{aligned} \quad (12)$$

Given a HO-MR-CBF, define the set of all control inputs that satisfy (12) as:

$$\begin{aligned} \mathbb{K}_{\text{HO-MR-CBF}} = \{\mathbf{u} \in \mathbb{U} : L_f^m h(\hat{\mathbf{x}}) + L_g L_f^{m-1} h(\hat{\mathbf{x}})\mathbf{u} + \\ O(h(\hat{\mathbf{x}})) + \alpha_m(\psi_{m-1}) - (\sum_{i=1}^{m} a_i(\mathbf{z}) + b(\mathbf{z})\|\mathbf{u}\|_2) > 0\}. \end{aligned} \quad (13)$$

With this definition, we now prove the safety of system (1) for the safety set $\mathcal{S}$ given the existence of a time-invariant HO-MR-CBF $h(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ under a measurement uncertainty of the form (6).

Proposition 1: Let the safe set $\mathcal{S}$ be defined as (2) by the continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. Assume the functions $L_f^m h : \mathbb{R}^n \to \mathbb{R}$, $L_g L_f^{m-1} h : \mathbb{R}^n \to \mathbb{R}$, $\alpha_m \circ \psi_{m-1}$, and $L_f^i(\alpha_{i-1} \circ \psi_{m-i-1}) : \mathbb{R}^n \to \mathbb{R}, \forall i \in \{1, 2, \ldots, m-1\}$, are continuously differentiable with Lipschitz constants $\bar{L}_{L_f^m h}$, $\bar{L}_{L_g L_f^{m-1} h}$, $\bar{L}_{\alpha_m \circ \psi_{m-1}}$, and $\bar{L}_{L_f^i(\alpha_{i-1} \circ \psi_{m-i-1})}, \forall i \in \{1, 2, \ldots, m-1\}$, respectively. Additionally, assume that the measurement error function $\epsilon(\mathbf{z})$ is of the form (6). If h is a HO-MR-CBF for system (1) on $\mathcal{S}$ with parameter functions $(\epsilon(\mathbf{z})(\bar{L}_{L_f^m h} + \sum_{i=1}^{m-1} \bar{L}_{L_f^i(\alpha_i \circ \psi_{m-i-1})} + \bar{L}_{\alpha_m \circ \psi_{m-1}}), \epsilon(\mathbf{z})(\bar{L}_{L_g L_f^{m-1} h}))$, then any locally Lipschitz continuous controller $k_{\text{safe}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, such that $k_{\text{safe}}(\mathbf{z}, \hat{\mathbf{x}}) \in \mathbb{K}_{\text{HO-MR-CBF}}(\mathbf{z}, \hat{\mathbf{x}}), \forall (\mathbf{z}, \hat{\mathbf{x}}) \in \hat{v}(\mathcal{S})$, renders (1) safe with respect to $\mathcal{S}$.

Proof: Our proof is similar to that of [19]. First define $c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ as:

$$c(\mathbf{x}, \mathbf{u}) = L_f^m h(\mathbf{x}) + L_g L_f^{m-1} h(\mathbf{x})\mathbf{u} + O(h(\mathbf{x})) + \alpha_m(\psi_{m-1})$$

Now, we need to show that for any $\mathbf{x} \in \mathcal{S}$ and $(\mathbf{z}, \hat{\mathbf{x}}) \in \hat{v}(\mathcal{S})$, $c(\mathbf{x}, \mathbf{u}) \geq 0$. A sufficient condition for this to hold is

$$\inf_{x \in \mathcal{X}(\mathbf{z})} c(\mathbf{x}, \mathbf{u}) \geq 0,$$

where $\mathcal{X}(\mathbf{z}) := \{\mathbf{x} \in \mathbb{R}^n : \exists\ \mathbf{e} \in \mathcal{E}(\mathbf{z}) \text{ s.t. } \hat{\mathbf{x}} = \mathbf{x} + \mathbf{e}(\mathbf{x})\}$ is the set of all possible *actual* states given the known measurement-estimate pair $(\mathbf{z}, \hat{\mathbf{x}})$. Now, from (5):

$$\begin{aligned}\inf_{x \in \mathcal{X}(\mathbf{z})} c(\mathbf{x}, \mathbf{u}) &= \inf_{e \in \mathcal{E}(\mathbf{z})} c(\hat{\mathbf{x}} - \mathbf{e}, \mathbf{u}) \\ &= c(\hat{\mathbf{x}}, \mathbf{u}) + \inf_{e \in \mathcal{E}(\mathbf{z})} c(\hat{\mathbf{x}} - \mathbf{e}, \mathbf{u}) - c(\hat{\mathbf{x}}, \mathbf{u}) \\ &= c(\hat{\mathbf{x}}, \mathbf{u}) - \sup_{e \in \mathcal{E}(\mathbf{z})} |c(\hat{\mathbf{x}} - \mathbf{e}, \mathbf{u}) - c(\hat{\mathbf{x}}, \mathbf{u})|.\end{aligned}$$

Now, consider $|c(\mathbf{x}', \mathbf{u}) - c(\mathbf{x}, \mathbf{u})| = |L_f^m h(\mathbf{x}') + L_g L_f^{m-1} h(\mathbf{x}')\mathbf{u} + O(h(\mathbf{x}')) + \alpha_m(\psi_{m-1}) - L_f^m h(\mathbf{x}) + L_g L_f^{m-1} h(\mathbf{x})\mathbf{u} + O(h(\mathbf{x})) + \alpha_m(\psi_{m-1})|$. Using the stated Lipschitz assumptions, the error bound defined by (6), and the triangle inequality, we get:

$$\begin{aligned}|c(\mathbf{x}', \mathbf{u}) - c(\mathbf{x}, \mathbf{u})| \leq (\bar{L}_{L_f^m h} + \sum_{i=1}^{m-1} \bar{L}_{L_f^i(\alpha_{i-1} \circ \psi_{m-i-1})} + \\ \bar{L}_{\alpha_m \circ \psi_{m-1}}) + \bar{L}_{L_g L_f^{m-1} h} \|\mathbf{u}\|_2)\epsilon(\mathbf{z}).\end{aligned}$$

Hence,

$$\begin{aligned}\inf_{x \in \mathcal{X}(\mathbf{z})} c(\mathbf{x}, \mathbf{u}) \geq c(\hat{\mathbf{x}}, \mathbf{u}) - (\bar{L}_{L_f^m h} + \sum_{i=1}^{m-1} \bar{L}_{L_f^i(\alpha_{i-1} \circ \psi_{m-i-1})} \\ + \bar{L}_{\alpha_m \circ \psi_{m-1}}) + \bar{L}_{L_g L_f^{m-1} h} \|\mathbf{u}\|_2)\epsilon(\mathbf{z}).\end{aligned}$$

By Definition 7, and choosing $\mathbf{u}$ from (13), we get $\inf_{x \in \mathcal{X}(\mathbf{z})} c(\mathbf{x}, \mathbf{u}) \geq 0$. ■

We next draw some comparisons between the existing measurement robust MR-CBF, and our proposed *high-order* variant HO-MR-CBF. First note that we can recover the existing definition of MR-CBF in [19] by setting $m = 1$ in our characterization of HO-MR-CBF; this is similar to the relation between CBF [2] and HO-CBF [11]. In addition, by comparing the definitions of (7) for the MR-CBF with (12) for HO-MR-CBF, we observe that the last (negative) term, which appears in the choice of our controller due to the state measurement errors, involves considerably *more* terms, related to the m class $\mathcal{K}$ functions a_i of the HO-MR-CBF. Intuitively, this larger negative term indicates a larger sensitivity of the choice of safe controllers to measurement errors, requiring the choice of the controller to be more conservative (i.e, have smaller $\|\mathbf{u}\|_2$) compared to the case without measurement errors.

Under the assumption that the obtained controllers are Lipschitz continuous, the control input can be generated by solving the following optimization problem:

$$\begin{aligned}&K_{\text{safe}}(\hat{\mathbf{x}}) = \operatorname*{argmin}_{\mathbf{u} \in \mathbb{U}} \frac{1}{2}\|\mathbf{u} - K_{\text{perf}}(\hat{\mathbf{x}})\|_2 \\ &\text{ s.t. } L_f^m h(\hat{\mathbf{x}}) + L_g L_f^{m-1} h(\hat{\mathbf{x}})\mathbf{u} + O(h(\hat{\mathbf{x}})) + \alpha_m(\psi_{m-1}) \\ &- (\bar{L}_{L_f^m h} + \sum_{i=1}^{m-1} \bar{L}_{L_f^i(\alpha_{i-1} \circ \psi_{m-i-1})} + \bar{L}_{\alpha_m \circ \psi_{m-1}}) + \\ &\bar{L}_{L_g L_f^{m-1} h} \|\mathbf{u}\|_2)\epsilon(\mathbf{z}) \geq 0\end{aligned} \tag{HO-MR-CBF-OP}$$

In the remainder of this section, we identify conditions on estimation errors under which the above problem is feasible. We further extend the problem above by introducing an additional constraint: bounded control inputs.

B. Introducing constrained control inputs

In the previous section, we provided the conditions under which a locally Lipschitz controller $k_{\text{safe}}(\mathbf{z}, \hat{\mathbf{x}}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ makes the system safe with respect to a safe set $\mathcal{S}$ defined as (2). Specifically, we show that if $k_{\text{safe}} \in \mathbb{K}_{\text{HO-MR-CBF}}$, then the system (1) is safe with respect to set $\mathcal{S}$, and control inputs to ensure this can be obtained by solving the optimization problem (HO-MR-CBF-OP).

Nonetheless, in many systems, the control inputs are restricted (e.g., by physical limitations), and this may further conflict with the constraint (12) imposed to ensure safety in (HO-MR-CBF-OP). Motivated by this, in this section, we further impose a constraint of the form $\mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max}$ on the controller choice. We provide conditions on the error function $e(\mathbf{z})$ under which such extended (HO-MR-CBF-OP) will be feasible under both safety and bounded controller constraints, and can therefore be solved to find safe controllers within the restricted control set.

To proceed, we begin by noting that typically, the sensor model $s(\mathbf{x})$ can be known to the designer, along with the estimation error bounds $\mathcal{E}(\mathbf{z})$; e.g., this is the case when using a Kalamn Filter with a $\mathcal{X}^2$ anomaly detector. Further, while the safe-set $\mathcal{S}$ is defined on the *actual* state $\mathbf{x}$, the controller only has access to the measurement-estimate pair $(\mathbf{z}, \hat{\mathbf{x}})$. Hence, similar to [19], to prove the feasibility of (HO-MR-CBF-OP) for all $\mathbf{x} \in \mathcal{S}$, we use the following definition:

$$\hat{\mathcal{X}}(\mathbf{x}) := \{\hat{\mathbf{x}} \in \hat{\mathbb{X}} \subset \mathbb{R}^n : \exists\ \mathbf{e} \in \mathcal{E}(s(\mathbf{x})) \text{ s.t. } \hat{\mathbf{x}} = \mathbf{x} + \mathbf{e}\}. \tag{14}$$

This is the set of all possible state estimates given a particular sensor model $s(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ and actual state $\mathbf{x} \in \mathbb{X}$. Also similar to [19], we extend the definition of $\epsilon(\mathbf{z})$ to $\epsilon(\mathbf{z}) = \epsilon(s(\mathbf{x})) =: \epsilon(\mathbf{x})$. Additionally, we define the following terms:

$$\begin{aligned}
\bar{\text{U}}_{\text{max}} &:= \sup_{\hat{\mathbf{x}}\in\hat{\mathcal{X}}} \|\mathbf{u}_{\text{max}}\|_2 \Big(\|L_g L_f^{m-1} h(\hat{\mathbf{x}})\|_2 - b(\mathbf{z}) \Big) \\
\bar{\text{U}}_{\text{min}} &:= \sup_{\hat{\mathbf{x}}\in\hat{\mathcal{X}}} \|\mathbf{u}_{\text{min}}\|_2 \Big(\|L_g L_f^{m-1} h(\hat{\mathbf{x}})\|_2 - b(\mathbf{z}) \Big) \\
\bar{\text{F}}_{\text{max}} &:= \sup_{\hat{\mathbf{x}}\in\hat{\mathcal{X}}} -L_f^m h(\hat{\mathbf{x}}) - \alpha_m(\psi_{m-1}(\hat{\mathbf{x}})) - O(h(\hat{\mathbf{x}}))
\end{aligned} \tag{15}$$

Given these definitions, we are ready to state our main result on a measurement errors upperbound under which the extended (12) will be a feasible optimization problem.

Proposition 2: Assume

$$\epsilon(\mathbf{x}) < \max\Big\{ \frac{\|\mathbf{u}_{\text{max}}\|_2 \|L_g L_f^{m-1} h(\mathbf{x})\|_2 + L_f^m h(\mathbf{x}) + \alpha_m(\psi_{m-1}(\mathbf{x})) + O(h(\mathbf{x}))}{2(\bar{\text{L}}_{\text{sum}} + \|\mathbf{u}_{\text{max}}\|_2 \bar{\text{L}}_{L_g L_f^{m-1} h})}, \frac{\|\mathbf{u}_{\text{min}}\|_2 \|L_g L_f^{m-1} h(\mathbf{x})\|_2 + L_f^m h(\mathbf{x}) + \alpha_m(\psi_{m-1}(\mathbf{x})) + O(h(\mathbf{x}))}{2(\bar{\text{L}}_{\text{sum}} + \|\mathbf{u}_{\text{min}}\|_2 \bar{\text{L}}_{L_g L_f^{m-1} h})} \Big\}$$

where $\bar{\text{L}}_{\text{sum}} = \bar{L}_{L_f^m h} + \sum_{i=1}^{m-1} \bar{L}_{L_f^i(\alpha_{i-1}\circ\psi_{m-i-1})} + \bar{L}_{\alpha_m\circ\psi_{m-1}}$, and the remaining terms follow the definitions in Proposition 1. Then, (12) with the added control input bounds of the form $\mathbf{u}_{\text{min}} \le \mathbf{u} \le \mathbf{u}_{\text{max}}$ is feasible $\forall \mathbf{x} \in \mathcal{S}$.

Proof: We begin by restating that if h is a HO-MR-CBF:

$$\begin{aligned}
&\sup_{\mathbf{u}\in\mathbb{U}} L_g L_f^{m-1} h(\hat{\mathbf{x}})\mathbf{u} - b(\mathbf{z})\|\mathbf{u}\|_2 > \\
&\quad - L_f^m h(\hat{\mathbf{x}}) - O(h(\hat{\mathbf{x}})) - \alpha_m(\psi_{m-1}) + \sum_{i=1}^{m} a_i(\mathbf{z})
\end{aligned} \tag{16}$$

Similar to [19], we use the change of variables $r = \|\mathbf{u}\|_2$ and $j = \mathbf{u}/\|\mathbf{u}\|_2$,

$$\begin{aligned}
&\sup_{\mathbf{u}\in\mathbb{U}} L_g L_f^{m-1} h(\hat{\mathbf{x}})\mathbf{u} - b(\mathbf{z})\|\mathbf{u}\|_2 \\
&= \sup_{\|\mathbf{u}_{\text{min}}\|_2 \le r \le \|\mathbf{u}_{\text{max}}\|_2} r\Big(\max_{\|j\|_2=1} L_g L_f^{m-1} j - b(\mathbf{z}) \Big) \\
&= \sup_{\|\mathbf{u}_{\text{min}}\|_2 \le r \le \|\mathbf{u}_{\text{max}}\|_2} r\Big(\|L_g L_f^{m-1} h(\hat{\mathbf{x}})\|_2 - b(\mathbf{z}) \Big)
\end{aligned}$$

We observe that (16) is feasible if

$$\sup_{\hat{\mathbf{x}}\in\hat{\mathcal{X}}, \mathbf{u}\in\mathbb{U}} \|\mathbf{u}\|_2 \Big(\|L_g L_f^{m-1} h(\hat{\mathbf{x}})\|_2 - b(\mathbf{z}) \Big) > \bar{\text{F}}_{\text{max}} + \sum_{i=1}^{m} a_i(\mathbf{z})$$

We first consider $\bar{\text{F}}_{\text{max}}$:

$$\begin{aligned}
\bar{\text{F}}_{\text{max}} &= \sup_{\hat{\mathbf{x}}\in\hat{\mathcal{X}}} -L_f^m h(\hat{\mathbf{x}}) - \alpha_m(\psi_{m-1}(\hat{\mathbf{x}})) - O(h(\hat{\mathbf{x}})) \\
&= -[\inf_{\hat{\mathbf{x}}\in\hat{\mathcal{X}}} L_f^m h(\hat{\mathbf{x}}) + \alpha_m(\psi_{m-1}(\hat{\mathbf{x}})) + O(h(\hat{\mathbf{x}}))] \\
&= -[\inf_{\hat{\mathbf{e}}\in\mathcal{E}(s(\mathbf{x}))} L_f^m h(\mathbf{x}+\mathbf{e}) + \alpha_m(\psi_{m-1}(\mathbf{x}+\mathbf{e})) + \\
&\quad O(h(\mathbf{x}+\mathbf{e}))] \\
&= -[L_f^m h(\mathbf{x}) + \inf_{\hat{\mathbf{e}}\in\mathcal{E}(s(\mathbf{x}))} L_f^m h(\mathbf{x}+\mathbf{e}) - L_f^m h(\mathbf{x}) + \\
&\quad \alpha_m(\psi_{m-1}(\mathbf{x})) + \inf_{\hat{\mathbf{e}}\in\mathcal{E}(s(\mathbf{x}))} \alpha_m(\psi_{m-1}(\mathbf{x}+\mathbf{e})) - \\
&\quad \alpha_m(\psi_{m-1}(\mathbf{x})) + O(h(\mathbf{x})) + \\
&\quad \inf_{\hat{\mathbf{e}}\in\mathcal{E}(s(\mathbf{x}))} O(h(\mathbf{x}+\mathbf{e})) - O(h(\mathbf{x}))] \\
&\le -[L_f^m h(\mathbf{x}) - \sup_{\hat{\mathbf{e}}\in\mathcal{E}(s(\mathbf{x}))} |L_f^m h(\mathbf{x}+\mathbf{e}) - L_f^m h(\mathbf{x})| + \\
&\quad \alpha_m(\psi_{m-1}(\mathbf{x})) - \sup_{\hat{\mathbf{e}}\in\mathcal{E}(s(\mathbf{x}))} |\alpha_m(\psi_{m-1}(\mathbf{x}+\mathbf{e})) - \\
&\quad \alpha_m(\psi_{m-1}(\mathbf{x}))| + O(h(\mathbf{x})) - \\
&\quad \sup_{\hat{\mathbf{e}}\in\mathcal{E}(s(\mathbf{x}))} |O(h(\mathbf{x}+\mathbf{e})) - O(h(\mathbf{x}))|] \\
&\le -[L_f^m h(\mathbf{x}) + \alpha_m(\psi_{m-1}(\mathbf{x})) + O(h(\mathbf{x})) - \bar{\text{L}}_{\text{sum}}\epsilon(\mathbf{x})]
\end{aligned}$$

Replacing the upper bound of $\bar{\text{F}}_{\text{max}}$, (12) is valid if:

$$\begin{aligned}
&\sup_{\hat{\mathbf{x}}\in\hat{\mathcal{X}}, \mathbf{u}\in\mathbb{U}} \|\mathbf{u}\|_2 \Big(\|L_g L_f^{m-1} h(\hat{\mathbf{x}})\|_2 - b(\mathbf{z}) \Big) > \\
&-L_f^m h(\mathbf{x}) - \alpha_m(\psi_{m-1}(\mathbf{x})) - O(h(\mathbf{x})) + 2\bar{\text{L}}_{\text{sum}}\epsilon(\mathbf{x})
\end{aligned}$$

When $\|L_g L_f^{m-1} h(\hat{\mathbf{x}})\|_2 - b(\mathbf{z}) > 0$, (16) is feasible if $\bar{\text{U}}_{\text{max}} > \bar{\text{F}}_{\text{max}}$. Whereas when $\|L_g L_f^{m-1} h(\hat{\mathbf{x}})\|_2 - b(\mathbf{z}) \le 0$, (16) is feasible if $\bar{\text{U}}_{\text{min}} > \bar{\text{F}}_{\text{max}}$. Replacing $(a_i(\mathbf{z}), b(\mathbf{z}))$ with their definitions from Proposition 1 and combining both inequalities, we get:

$$\epsilon(\mathbf{z}) = \epsilon(\mathbf{x}) < \max\Big\{ \frac{\|\mathbf{u}_{\text{max}}\|_2 \|L_g L_f^{m-1} h(\hat{\mathbf{x}})\|_2 + L_f^m h(\mathbf{x}) + \alpha_m(\psi_{m-1}(\mathbf{x})) + O(h(\mathbf{x}))}{2\bar{\text{L}}_{\text{sum}} + \|\mathbf{u}_{\text{max}}\|_2 \bar{\text{L}}_{L_g L_f^{m-1} h}}, \frac{\|\mathbf{u}_{\text{min}}\|_2 \|L_g L_f^{m-1} h(\hat{\mathbf{x}})\|_2 + L_f^m h(\mathbf{x}) + \alpha_m(\psi_{m-1}(\mathbf{x})) + O(h(\mathbf{x}))}{2\bar{\text{L}}_{\text{sum}} + \|\mathbf{u}_{\text{min}}\|_2 \bar{\text{L}}_{L_g L_f^{m-1} h}} \Big\}$$

Finally, similar to [19] we observe that:

$$\begin{aligned}
\|L_g L_f^{m-1} h(\mathbf{x})\|_2 &= \|L_g L_f^{m-1} h(\hat{\mathbf{x}} - \mathbf{e}(\mathbf{x}))\|_2 \\
\|L_g L_f^{m-1} h(\mathbf{x})\|_2 &\le \|L_g L_f^{m-1} h(\hat{\mathbf{x}})\|_2 + \bar{\text{L}}_{L_g L_f^{m-1}} \epsilon(\mathbf{x})
\end{aligned}$$

giving:

$$\epsilon(\mathbf{x}) < \max\Big\{ \frac{\|\mathbf{u}_{\text{max}}\|_2 \|L_g L_f^{m-1} h(\mathbf{x})\|_2 + L_f^m h(\mathbf{x}) + \alpha_m(\psi_{m-1}(\mathbf{x})) + O(h(\mathbf{x}))}{2(\bar{\text{L}}_{\text{sum}} + \|\mathbf{u}_{\text{max}}\|_2 \bar{\text{L}}_{L_g L_f^{m-1} h})}, \frac{\|\mathbf{u}_{\text{min}}\|_2 \|L_g L_f^{m-1} h(\mathbf{x})\|_2 + L_f^m h(\mathbf{x}) + \alpha_m(\psi_{m-1}(\mathbf{x})) + O(h(\mathbf{x}))}{2(\bar{\text{L}}_{\text{sum}} + \|\mathbf{u}_{\text{min}}\|_2 \bar{\text{L}}_{L_g L_f^{m-1} h})} \Big\}$$

■

In the limit when $\|\mathbf{u}_{\text{max}}\|_2 \to \infty$ and $\|\mathbf{u}_{\text{min}}\|_2 \to 0$ (i.e., when there are no constraints on the control inputs), the bound in

Proposition 2 reduces to:

$$\epsilon(\mathbf{x}) < \max\left\{\frac{\|L_g L_f^{m-1} h(\mathbf{x})\|_2}{2\mathfrak{L}_{L_g L_f^{m-1} h}}, \frac{L_f^m h(\mathbf{x}) + \alpha_m(\psi_{m-1}(\mathbf{x})) + O(h(\mathbf{x}))}{2\mathfrak{L}_{\text{sum}}}\right\}$$

As such, this proposition further extends the bounds obtained in [19] for the feasibility of the (unconstrained) MR-CBF controller choice, with the only difference being that the terms are now in their high-order variation for the feasibility of our (unconstrained) HO-MR-CBF.

IV. Numerical Example

In the previous section, we extended the MR-CBFs to *high-order* MR-CBF, further introducing bounded controller constraints, and identified bounds on the measurement error under which the resulting optimization problem for identifying safe controllers is feasible. In this section we showcase the approach on a problem where a robot is required to avoid an obstacle in presence of a constant bias in its measurements. The model of the robot is given by:

$$\begin{bmatrix} x \\ y \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \tag{17}$$

where the states $\mathbf{x} = [x, y, v, \theta]$ denote the x position coordinate, y position coordinate, velocity and heading angle, respectively, and the control inputs u_1 and u_2 indicate the steering rate and acceleration respectively. The control limits are set to 1 rad/s and 5 m/s^2, respectively. We assume a safety requirement where the robot needs to keep a certain distance D to the obstacle, given by:

$$h(\mathbf{x}) = (x - x_0)^2 + (y - y_o)^2 - D^2 \geq 0 \tag{18}$$

where (x_o, y_o) are the coordinates of the obstacle. Additionally, we assume there is a constant 1m bias in the position estimates, giving us an upper bound on the error $\epsilon(\mathbf{z}) = 1$, where $\mathbf{z}$ denote the position sensor measurements. We observe that $h(\mathbf{x})$ is indeed high-order with degree $m = 2$. An MPC controller without the knowledge of the safety requirement is used as the performance controller. The generated control signal is then passed through a high-order CBF based safety filter which outputs the final control signal input to system (17). The required Lipschitz constants are derived manually. We use the penalty method as introduced in [11] along with linear class $\mathcal{K}$ functions to ensure feasibility of the ensuing optimization. We use the same penalty value p for the two class $\mathcal{K}$ functions.

We first run this architecture with the safety filter built with the QP using the regular HOCBF constraint (11). We use a penalty $p = 1.5$. The resulting trajectory is illustrated in fig. 1. The boxes around the estimated state indicate the potential set in which the estimated state can lie. Since the HOCBF has access to only the estimated state, it can be seen that the estimated trajectory is safe whereas the true trajectory collides with the obstacle.

Next, we run the same simulation with the safety filter using the proposed measurement robust variation of the HOCBF as seen in (HO-MR-CBF-OP) using the same penalty as above. The resulting trajectory is shown in fig. 2. It is observed that neither the true state, nor the potential error set collides with the obstacle. The resulting trajectory depends on the choice of penalties and the type of class $\mathcal{K}$ functions (for example, linear v/s quadratic). The resulting trajectories for a few different penalty values are shown in fig. 3.

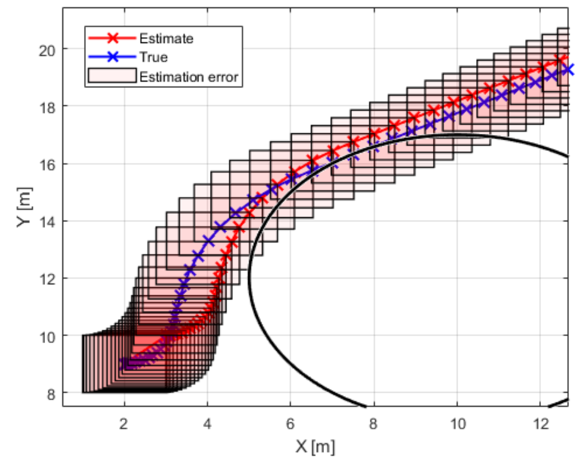


Fig. 1: Simulation using regular high-order CBF under measurement uncertainty. The true trajectory collides with the obstacle (black curve) while the estimate does not.

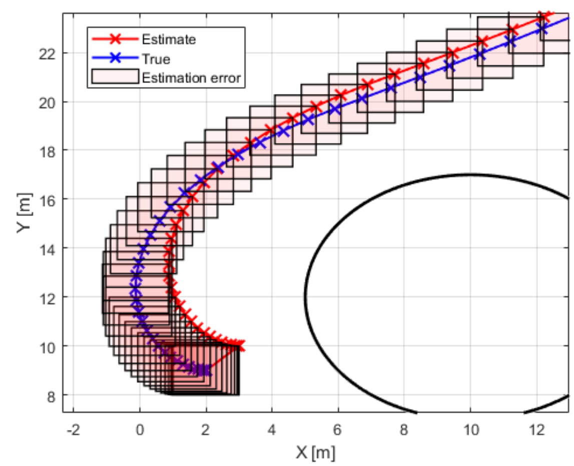


Fig. 2: Simulation using the proposed high-order measurement robust CBF under measurement uncertainty. The true trajectory does not collide with the obstacle.

A. Maximum error bound analysis

Next, we look at the upper bounds on the error $\epsilon(\mathbf{x})$ at different states for which (HO-MR-CBF-OP) is feasible. Figure 4a shows the upper bounds at $v = 10$ m/s and $\theta = 0$ deg, which indicates a situation where the robot is moving to the right at 10 m/s. The obstacle is the empty space centered at $(10, 12)$. It can be observed that the positions

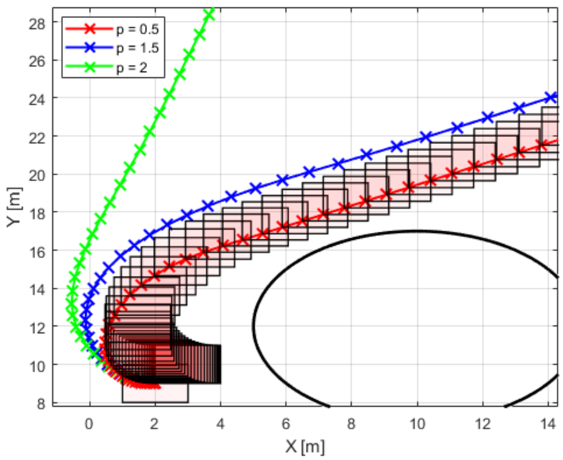
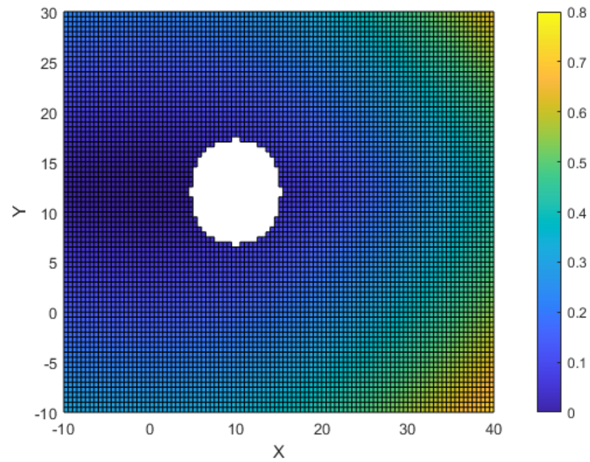


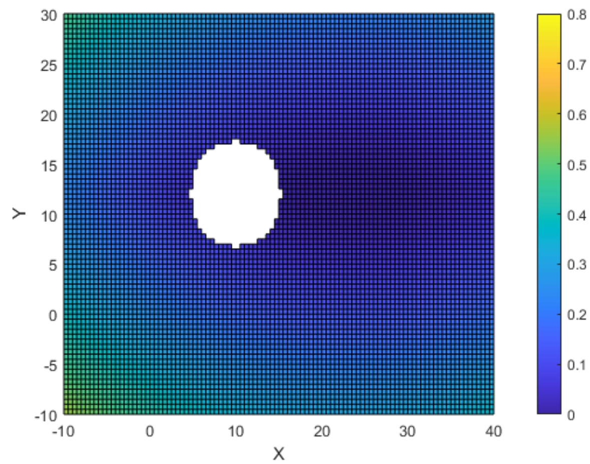
Fig. 3: Simulation using the proposed high-order measurement robust CBF under measurement uncertainty with different penalty values. The conservativness increases with increasing penalty. The error set is also shown as boxes, for reference.

to the left of the obstacle have a very low upper bound. This is because since the robot is moving to the right (towards the obstacle), the error bounds must be lower to ensure a safe control action is found, while farther away, a safe action can be found for higher error bounds as well. Figure 4b shows the same for $v = 10$m/s and $\theta = 180$ deg. Now, since the robot is moving left, the positions to the right of the obstacle have a lower upper bound.

Next, we look at the effect of increased control limits, shown in fig. 5. While intuitively a higher control action should allow for more flexibility in motion, we notice that the upper bound of the error reduces. This could be due to the sensitivity of $\epsilon(\mathbf{z})$ to $\|u\|_2$ as can be seen in (HO-MR-CBF-OP).



(a) at 10 m/s velocity and 0 deg heading



(b) at 10 m/s velocity and 180 deg heading

Fig. 4: Upper bound on error for feasible (HO-MR-CBF-OP) at different states

V. Conclusion

We studied the problem of safe control in the presence of state measurement errors and bounded controller constraints using High-Order Measurement Robust Control Barrier Functions (HO-MR-CBF). We first extended the formulation of the measurement robust control barrier function to their high-order variation to handle systems with high-relative degree. We then provided conditions on the bound of the error in the state estimates for which a HO-MR-CBF optimization problem is feasible. We applied the proposed HO-MR-CBF on a collision avoidance problem using a nonlinear kinematic model of a wheeled robot in the presence of state measurement bias. While the HOCBF fails to satisfy the safety requirement, we see that the proposed HO-MR-CBF does. Additionally, by varying the penalties for the class $\mathcal{K}$ functions used, we obtain multiple safe trajectories with varying degrees of conservativeness. Future work include extension of the proposed approach to discrete domains, and augmenting with classical state estimators such as Kalman Filters to provide stronger performance and safety guarantees. From an application viewpoint, we seek to further explore the applicability of this approach to provide safety guarantees for cyber-physical systems under stealthy attacks.

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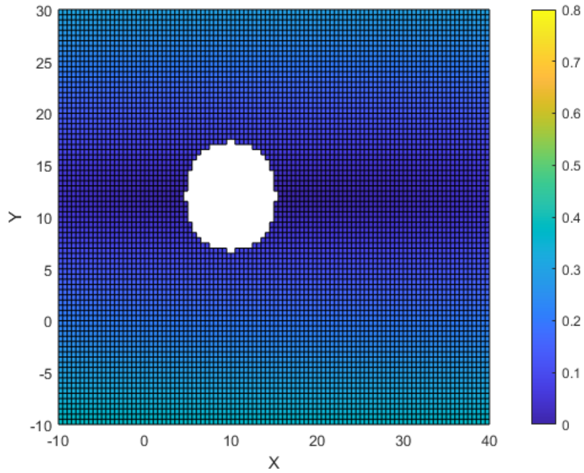

Fig. 5: Upper bounds on error for feasible (HO-MR-CBF-OP) with higher control limits.

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